Free Electron Lasers

Lecture II

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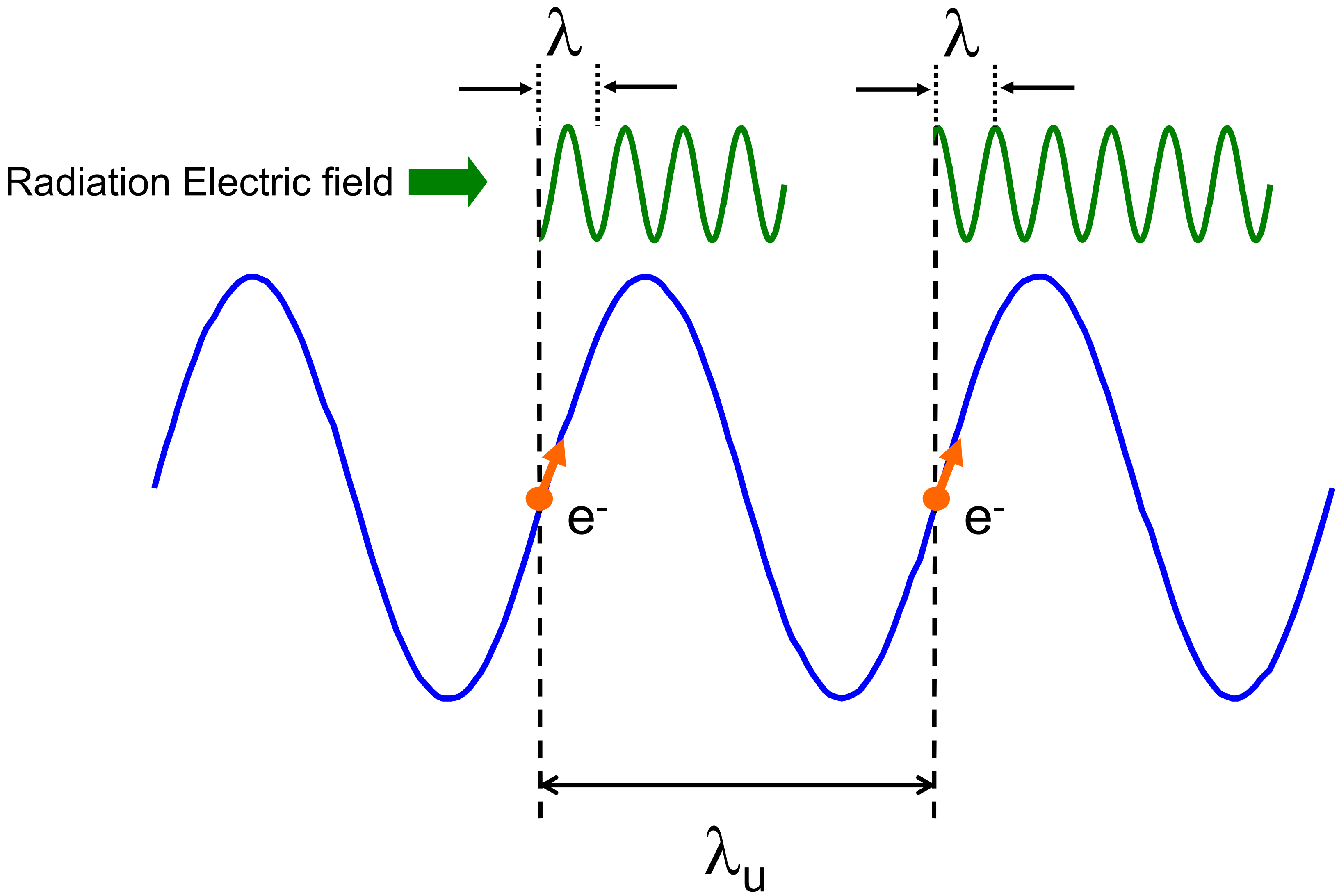
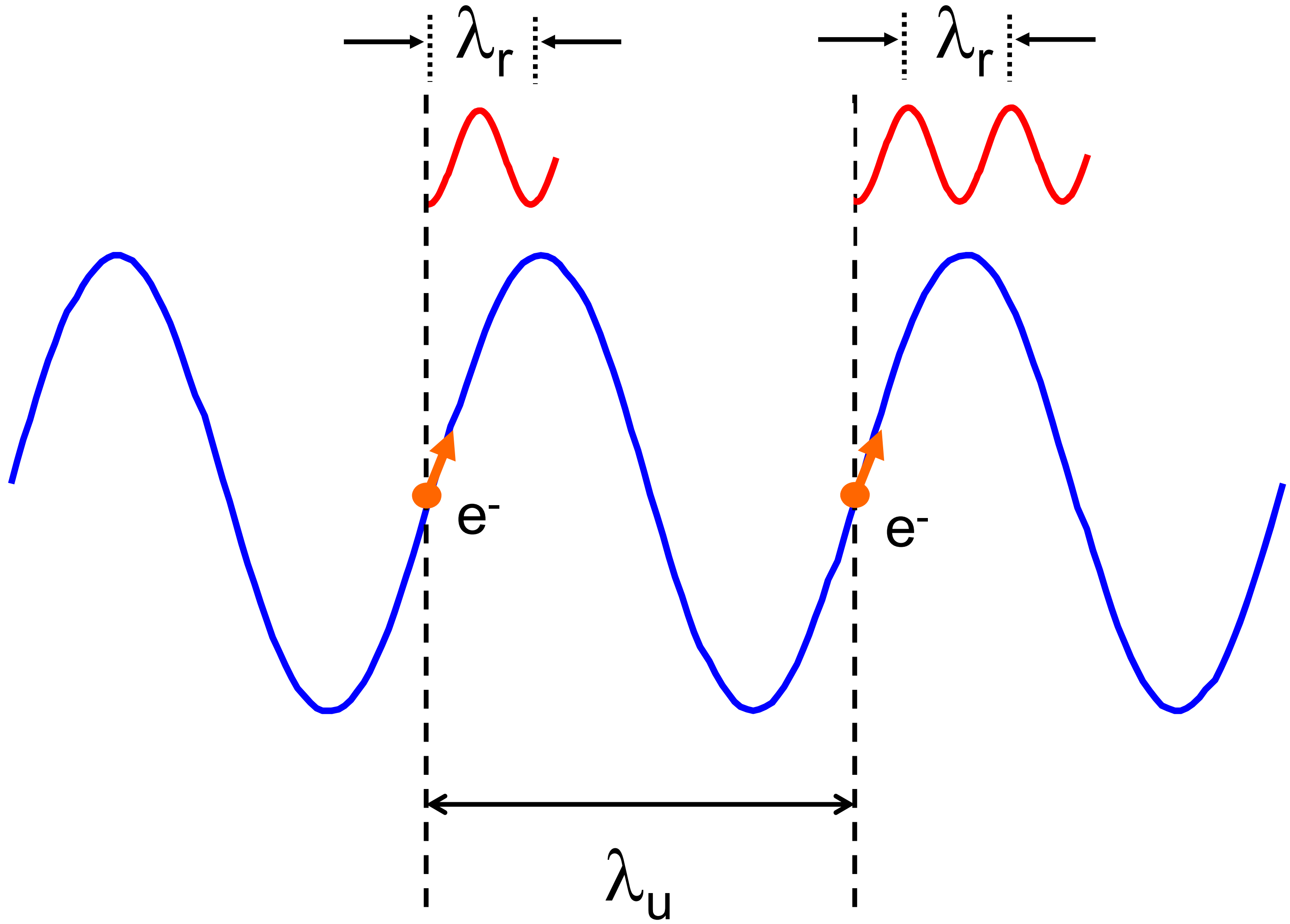
Radiation interference in an undulator and resonant emission

Non-resonant emission - destructive interference

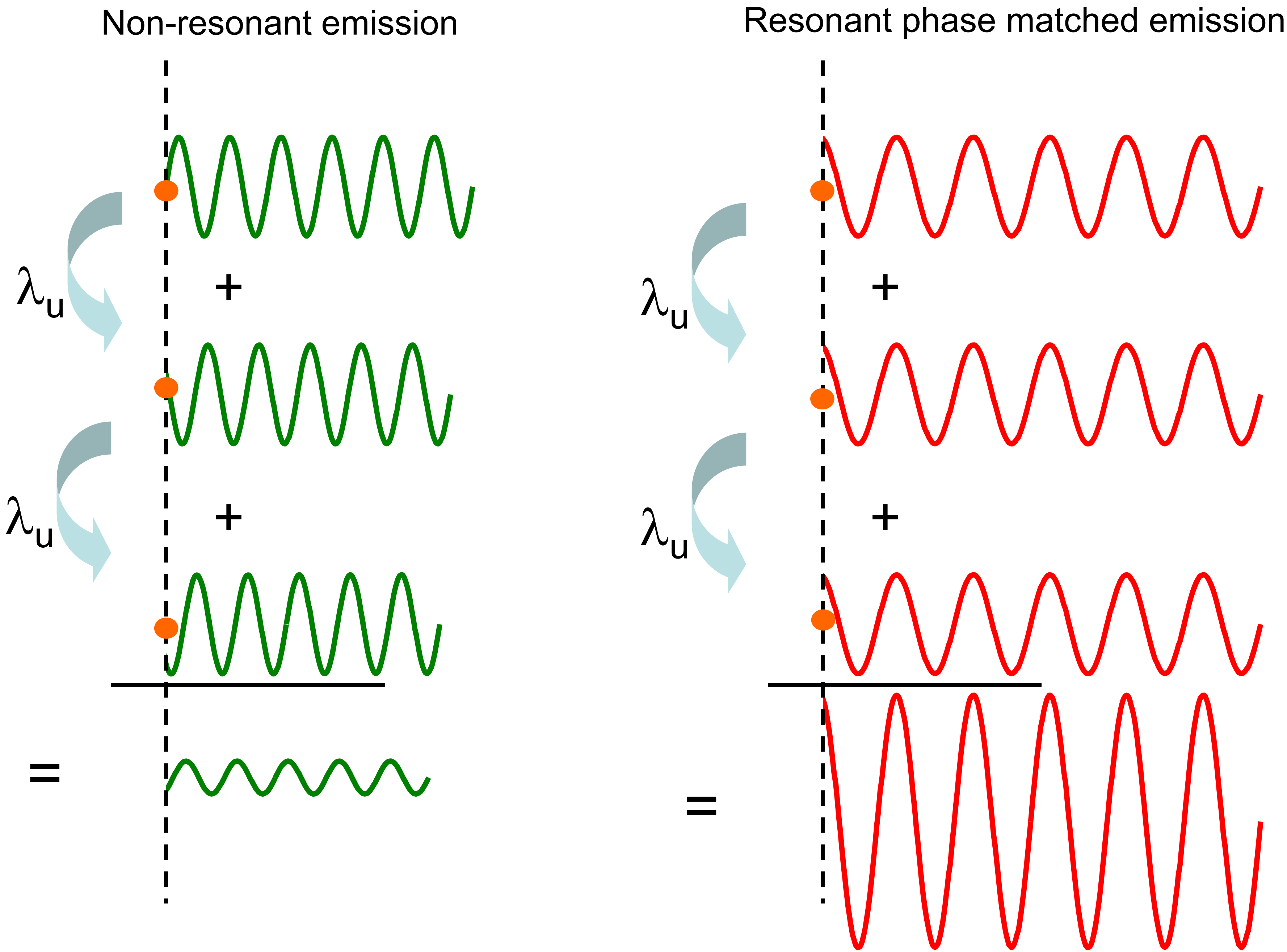


The radiation is ***not phase-matched*** to the electron trajectory.

Resonant emission - constructive interference

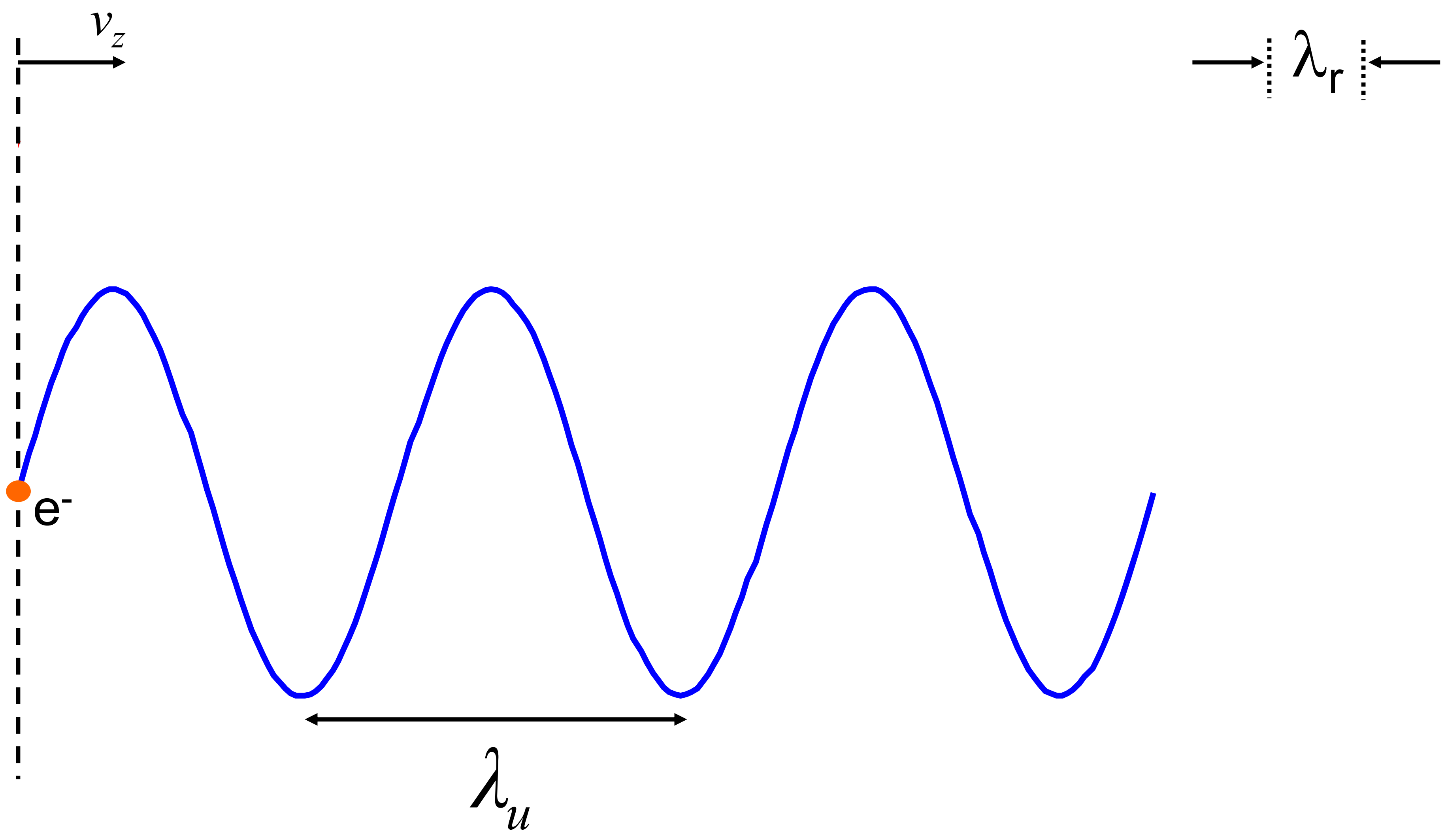


The radiation and electron trajectory are ***phase-matched***.

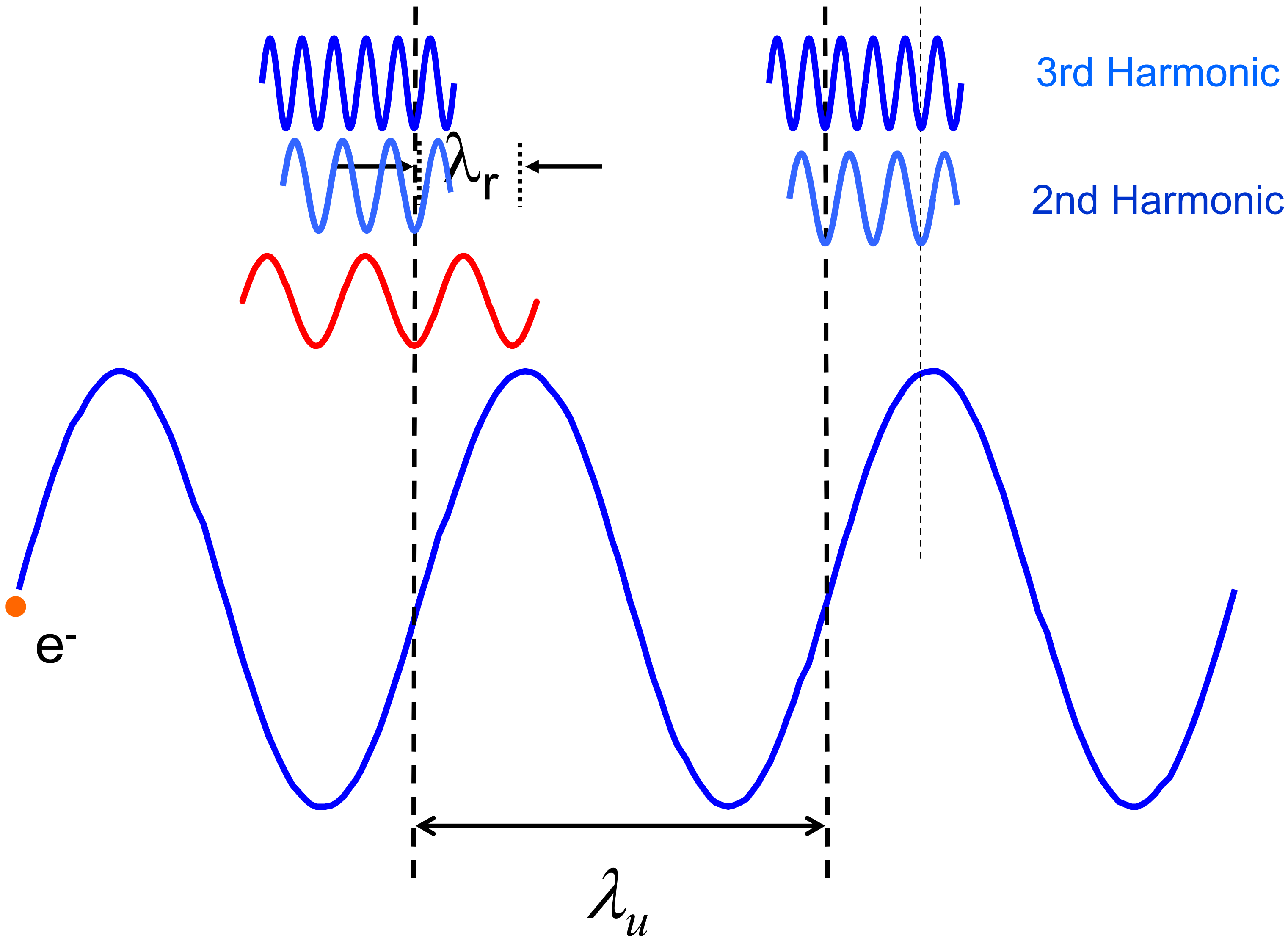


Non-resonant emission
Resonant phase matched emission
λu
+
λu
+
=
λu
+
λu
+
=

Resonant phase matched emission by an electron



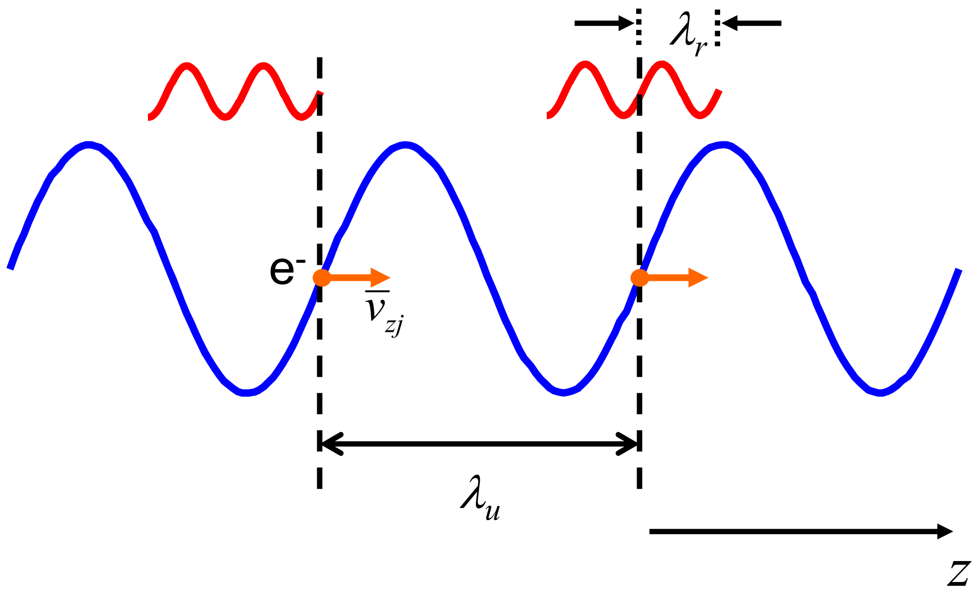
Resonant phase matched emission for harmonics



Harmonics of the fundamental are also phase-matched.

What are properties of radiation from an undulator ?

Resonant emission - constructive interference



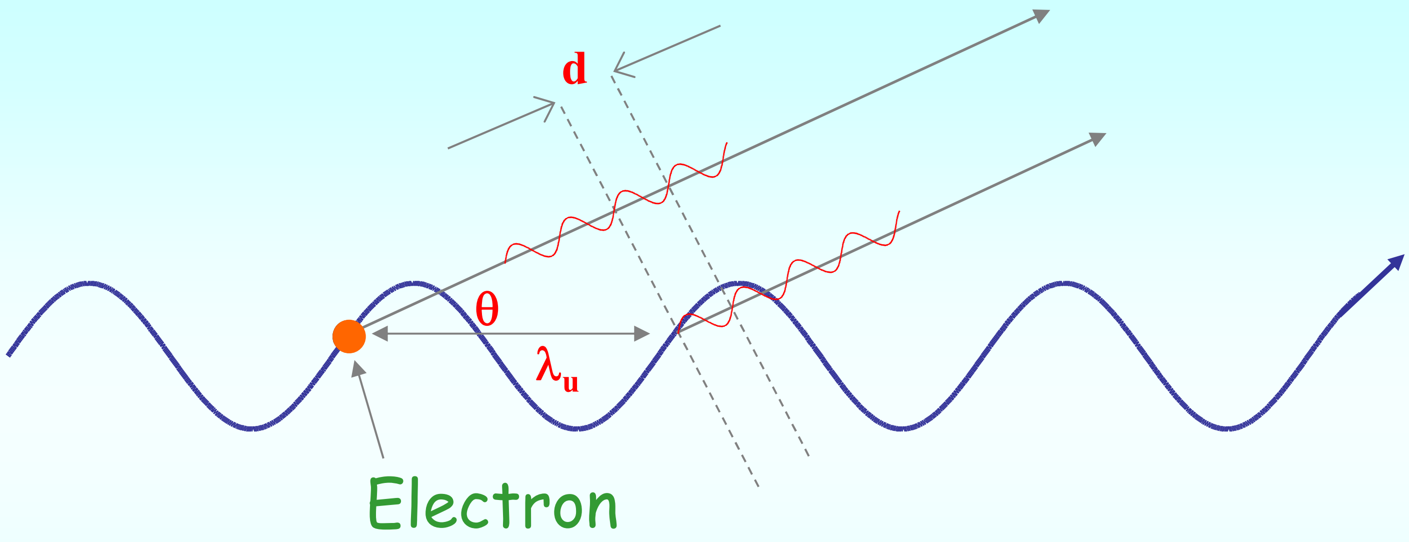
The time taken for the electron to travel one undulator period:

A resonant radiation wavefront will have travelled $\Rightarrow$

Equating:

Resonant emission - constructive interference including harmonics and angle from undulator axis

Condition for constructive interference: $$d = n\lambda = \frac{\lambda_u}{\beta_{zj}} - \lambda_u \cos\theta$$



Where: $n = 1, 2, 3, \ldots$ is an integer representing the harmonic number

Undulator Equation

Substituting in for the average longitudinal velocity of the electron, $\bar{\beta}_z$, for the earlier planar case:

Substitute $\bar{\beta}_{zj} \approx 1 - \frac{1}{2\gamma_0^2}\left(1 + \frac{a_u^2}{2}\right)$ into $n\lambda_r = \frac{\lambda_u}{\bar{\beta}_{zj}} - \lambda_u \cos\theta$

$$\Rightarrow \quad \lambda_r = \frac{\lambda_u}{2n\gamma_0^2}\left(1 + \bar{a}_u^2 + \theta^2\gamma_0^2\right)$$

Including angular dependence

$\bar{a}_u = \frac{e\lambda_u B_u^{RMS}}{2\pi mc}$ is the RMS "wiggler/undulator parameter"
- In this form also valid for helical undulators

For a 3 GeV electron passing through a 5 cm period undulator with $\bar{a}_u$ = 3, the wavelength of the first harmonic (n = 1) on axis ($\theta = 0$) is ~ 4 nm

The expression for the fundamental resonant wavelength shows us the origin of the FEL tunability:

$$\lambda_r = \left(\frac{1 + \overline{a}_u^{\,2}}{2\gamma_0^{\,2}} \right) \lambda_u$$

As the beam energy is increased, the spontaneous emission peak moves to shorter wavelengths.

For an undulator parameter $\overline{a}_u \approx 1$ and λ_u=1cm :

For mildly relativistic beams ($\gamma \approx 3$) : $\lambda_r \approx$ 1mm (microwaves)
more relativistic beams ($\gamma \approx 30$) : $\lambda_r \approx$ 10μm (infra-red)
ultra-relativistic beams ($\gamma \approx 30000$) : $\lambda_r \approx$ 0.1nm (X-ray)

Further tunability is possible through B_u and λ_u as $\overline{a}_u \equiv B_u \lambda_u$

Spontaneous undulator spectrum

We can calculate the spontaneous emission spectrum in the frequency range $d\omega$ and solid angle $d\Omega$ around the observation direction $\underset{\sim}{n}$ *by inserting the expression for the electron trajectory into the standard formula for far-field emission* from an accelerated charged particle (see e.g. “Classical Electrodynamics” by Jackson , ch. 14)

$$\frac{d^2 I}{d\omega\, d\Omega}(\underset{\sim}{n},\omega) = \frac{e^2\omega^2}{4\pi^2 c}\left| \int_{-\infty}^{\infty} \underset{\sim}{n}\times\left(\underset{\sim}{n}\times\underset{\sim}{\beta}\right) e^{i\omega\left(t-\underset{\sim}{n}\cdot\underset{\sim}{r}/c\right)}\, dt \right|^2$$

where $\underset{\sim}{r}(t)$ is the electron position at time t, and $\underset{\sim}{\beta} = \frac{\underset{\sim}{v}}{c}$

Spontaneous spectrum

If we do this we find that the spectrum on axis ($\underset{\sim}{n} = \hat{\underset{\sim}{z}}$) is :

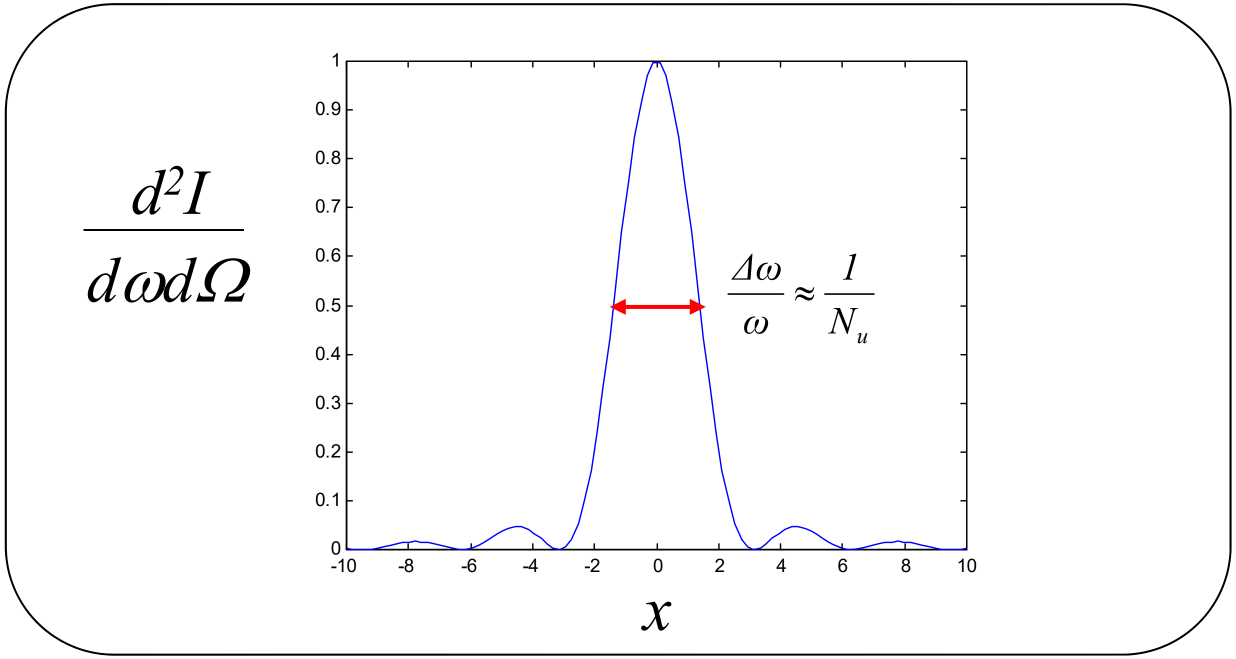
$$\frac{d^2 I}{d\omega\, d\Omega}(\underset{\sim}{n} = \hat{\underset{\sim}{z}}, \omega) \propto \frac{\sin^2 x}{x^2}$$

Where: $x = \pi N_u \dfrac{\omega - \omega_r}{\omega_r}$

N_u is the number of undulator periods

$\omega_r = \dfrac{2ck_u\gamma_0^2}{1+\bar{a}_u^2}$ is the central (resonance) frequency

On axis spontaneous spectrum therefore looks like :



Main features :

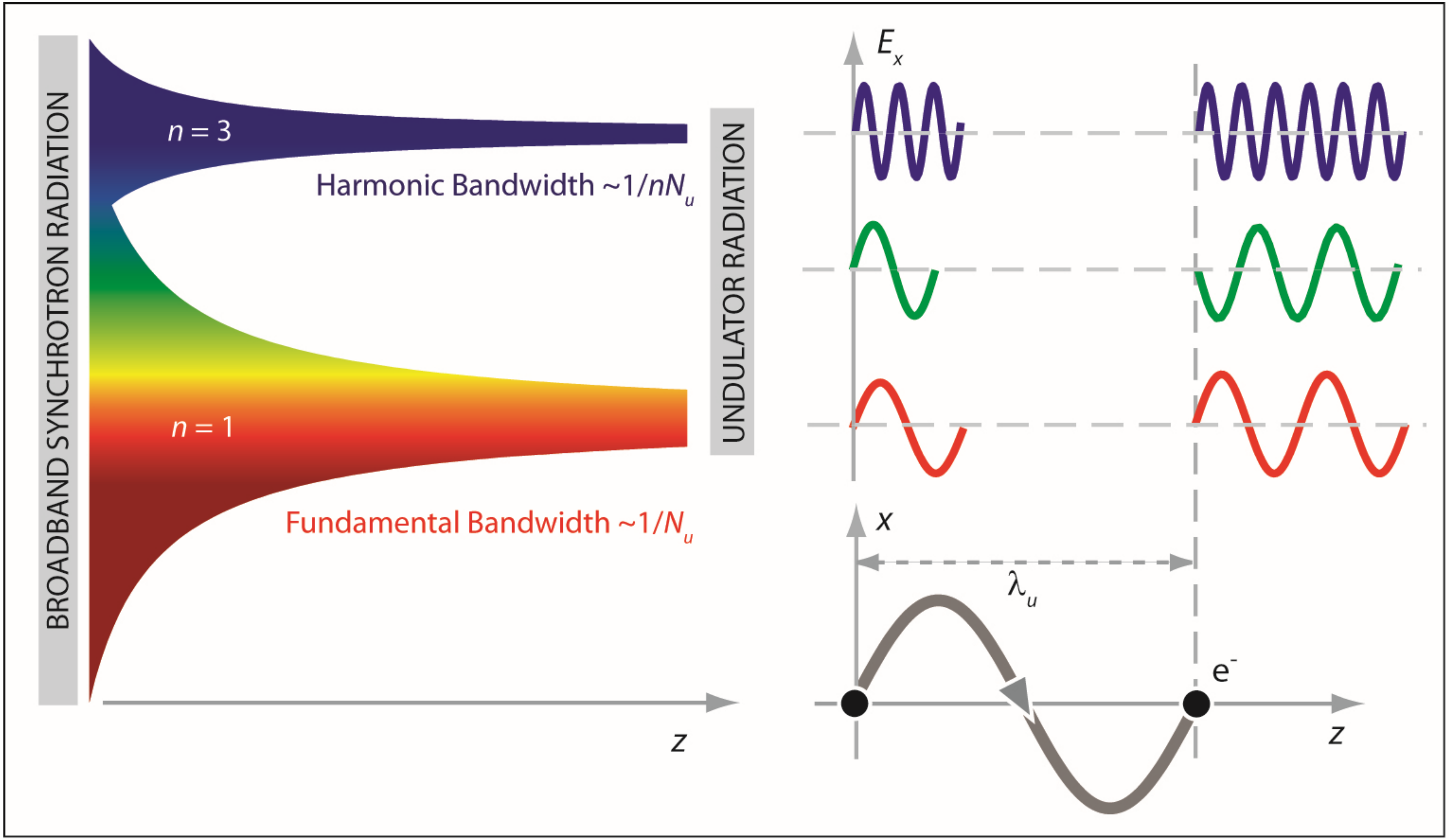
- Spectrum strongly peaked at frequency ω_r

  i.e. at wavelength

$$\lambda_r = \frac{2\pi c}{\omega_r} = \lambda_u \left(\frac{1 + \bar{a}_u^2}{2\gamma_0^2} \right)$$

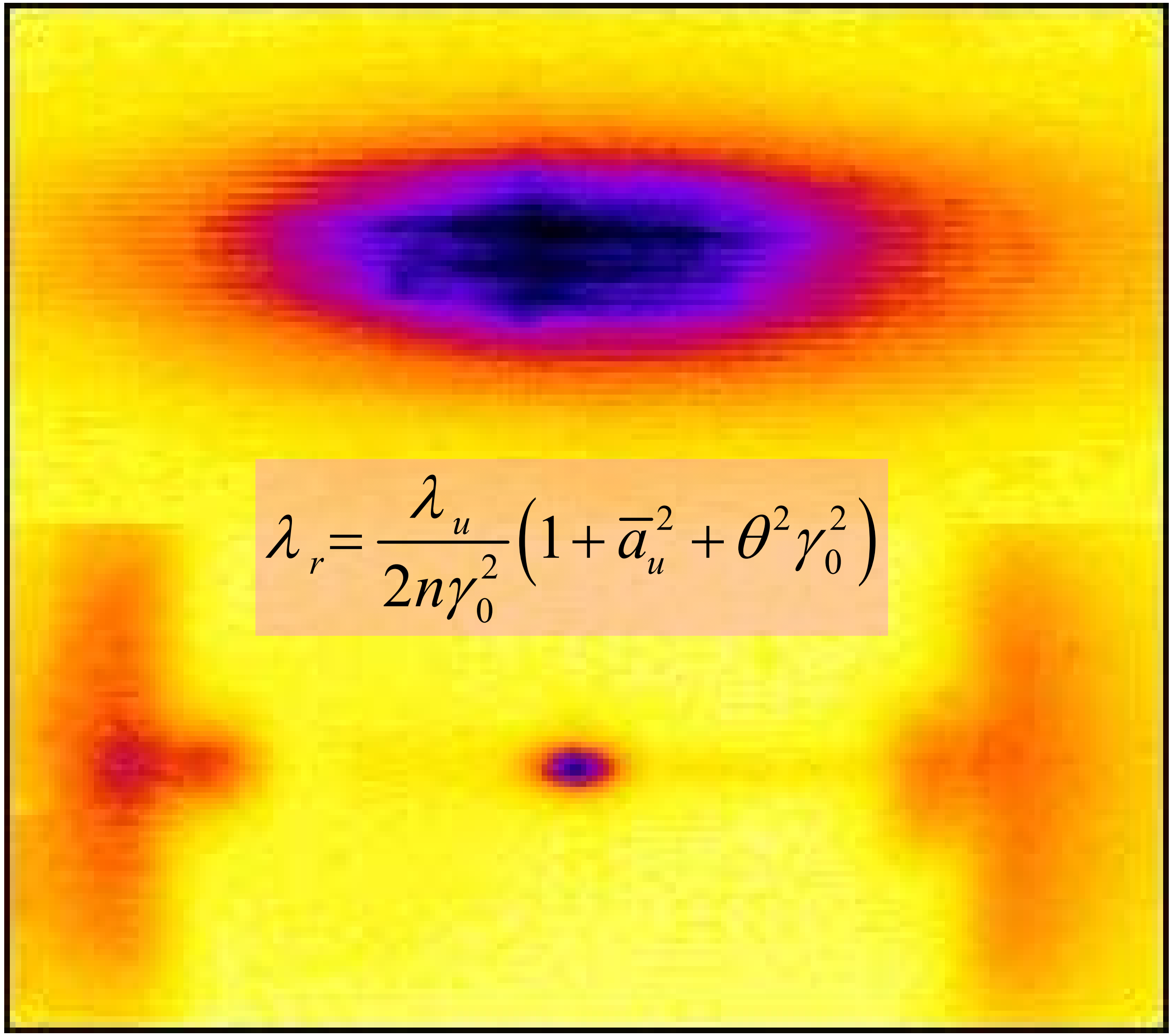
- Width of spectrum $\frac{\Delta\omega}{\omega} \approx \frac{1}{N_u}$

Summary Undulator Radiation I



Note 2nd harmonic not shown here

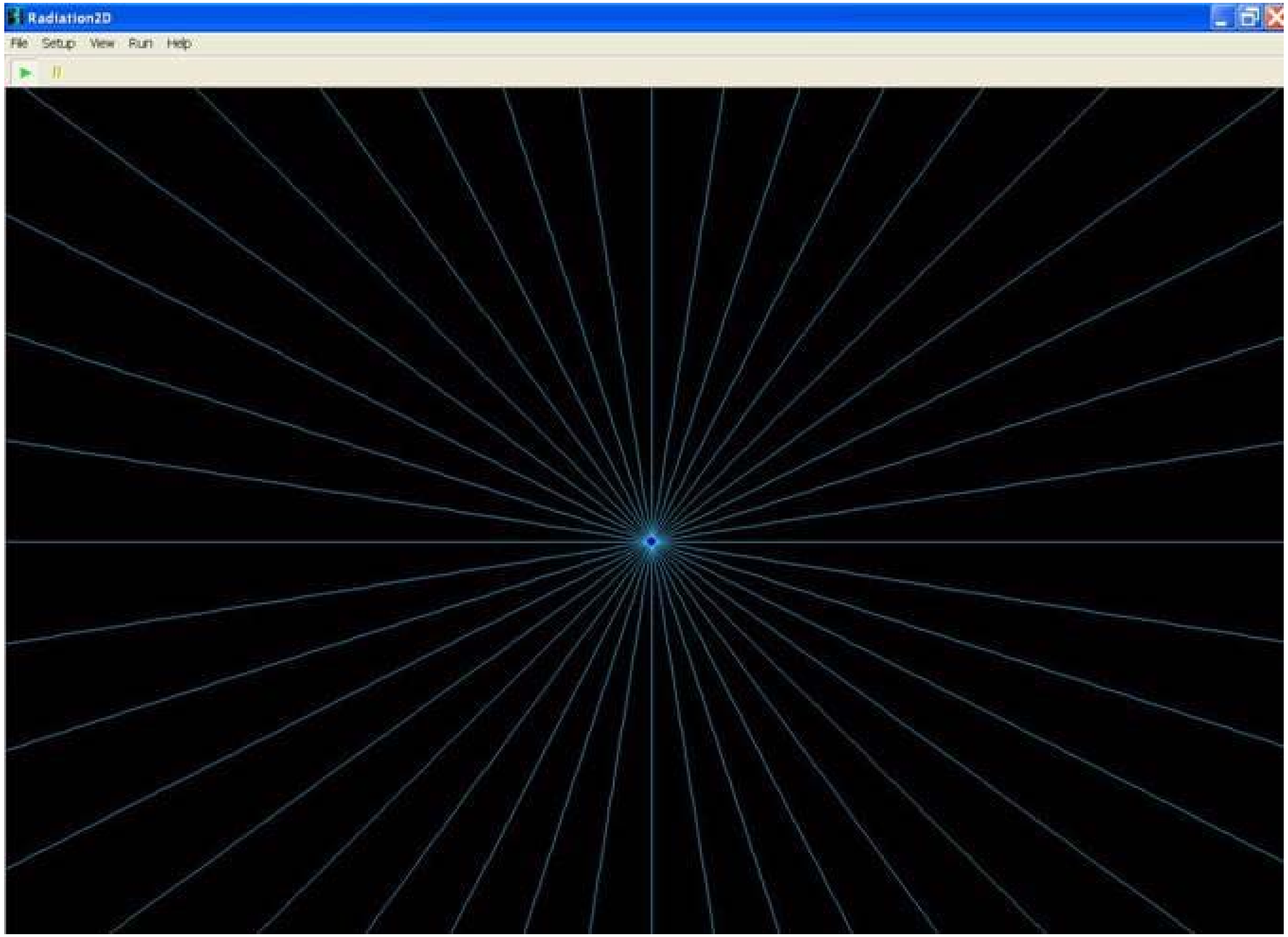
Summary Undulator Radiation II



$$\lambda_r = \frac{\lambda_u}{2n\gamma_0^2}\left(1 + \bar{a}_u^2 + \theta^2\gamma_0^2\right)$$

Undulator radiation (top) focused on a spot (bottom) by a refractive lens.

Undulator radiation
Setup>trajectory>undulator



Code available at: http://www.shintakelab.com/en/enEducationalSoft.htm

Electron bunching in a fixed radiation field

The electron-radiation interaction

The Lorentz force (electron dynamics)

$$\underset{\sim}{F}_j = q\left[\underset{\sim}{E} + \underset{\sim}{v}_j \times \underset{\sim}{B}\right]$$

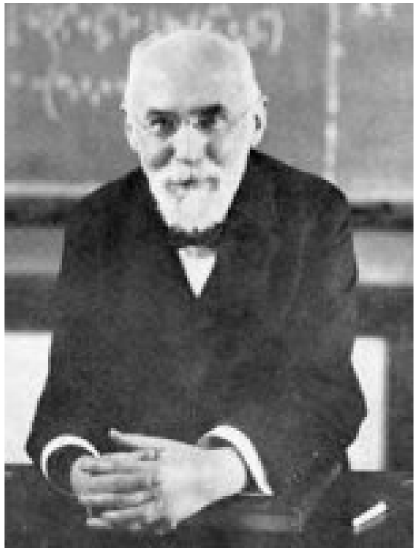
Maxwell wave equation* (radiation evolution)

$$\nabla^2 \underset{\sim}{E} - \frac{1}{c^2}\frac{\partial^2 \underset{\sim}{E}}{\partial t^2} = \mu_0 \frac{\partial \underset{\sim}{J}}{\partial t}$$

Both equations must be solved together simultaneously (self-consistently) to fully describe the FEL interaction

*Neglect static fields (space charge effects) – Compton limit

How the electron is effected by the resonant radiation



Hendrick Antoon Lorentz

The Lorentz Force Equation:

$$\underset{\sim}{F}_j = \frac{d\left(\gamma_j m_0 \underset{\sim}{v}_j\right)}{dt} = -|e|\left[\underset{\sim}{E} + \underset{\sim}{v}_j \times \underset{\sim}{B}\right]$$

Can calculate

The rate of change of electron energy

$$\frac{d\left(\gamma_j m_0 c^2\right)}{dt} = -|e|\,\underset{\sim}{E} \cdot \underset{\sim}{v}_j$$

Slow energy exchange

The rate of change of electron energy: $$\frac{d\left(\gamma_j m_0 c^2\right)}{dt} = -|e|\,\underset{\sim}{E} \cdot \underset{\sim}{v}_j$$

Consider plane-wave field: $\underset{\sim}{E} = \hat{\underset{\sim}{x}}\, E_0 \sin\left(k_r z - \omega_r t\right)$

Interacting with an electron on trajectory: $\beta_{xj} = -\frac{a_u}{\gamma_0}\cos\left(k_u z_j\right)$

Assuming: $\gamma_j \approx \gamma_0$

$$\Rightarrow \quad \frac{d\left(\gamma_j m_0 c^2\right)}{dt} = -|e|\,\underset{\sim}{E} \cdot \underset{\sim}{v}_j = -|e|\,E_0 \sin\left(k_r z_j - \omega_r t\right)\left(-\frac{a_u}{\gamma_0}\cos\left(k_u z_j\right)\right)$$

$$\Rightarrow \quad \frac{d\gamma_j}{dt} = \frac{|e|\,a_u E_0}{\gamma_0 m_0 c^2}\sin\left(k_r z_j - \omega_r t\right)\cos\left(k_u z_j\right)$$

$$= \frac{|e|\,a_u E_0}{\gamma_0 m_0 c^2}\frac{1}{2}\left(\sin\left(\left(k_r + k_u\right) z_j - \omega_r t\right) + \sin\left(\left(k_r - k_u\right) z_j - \omega_r t\right)\right)$$

Slow energy exchange

$$\frac{d\gamma_j}{dt} = \frac{|e| a_u E_0}{\gamma_0 m_0 c^2} \frac{1}{2}\left(\sin\left((k_r + k_u) z_j - \omega_r t\right) + \sin\left((k_r - k_u) z_j - \omega_r t\right)\right)$$

The first $\sin$ term on RHS is a wave with phase velocity in z direction of:

$$v_z = \frac{\omega_r}{k_r + k_u} = \frac{ck_r}{k_r + k_u} \quad \Rightarrow \quad \beta_z = \frac{k_r}{k_r + k_u} < 1$$

Recall previous result for resonance by considering phase-matching:

$$\lambda_r = \frac{1 - \bar{\beta}_{zj}}{\bar{\beta}_{zj}} \lambda_u \quad \Rightarrow \quad \bar{\beta}_{zj} = \frac{\lambda_u}{\lambda_r + \lambda_u} = \frac{k_r}{k_r + k_u}$$

So, a resonant electron with average speed $\bar{\beta}_{zj}$ will have $\frac{d\gamma_j}{dt} \approx$

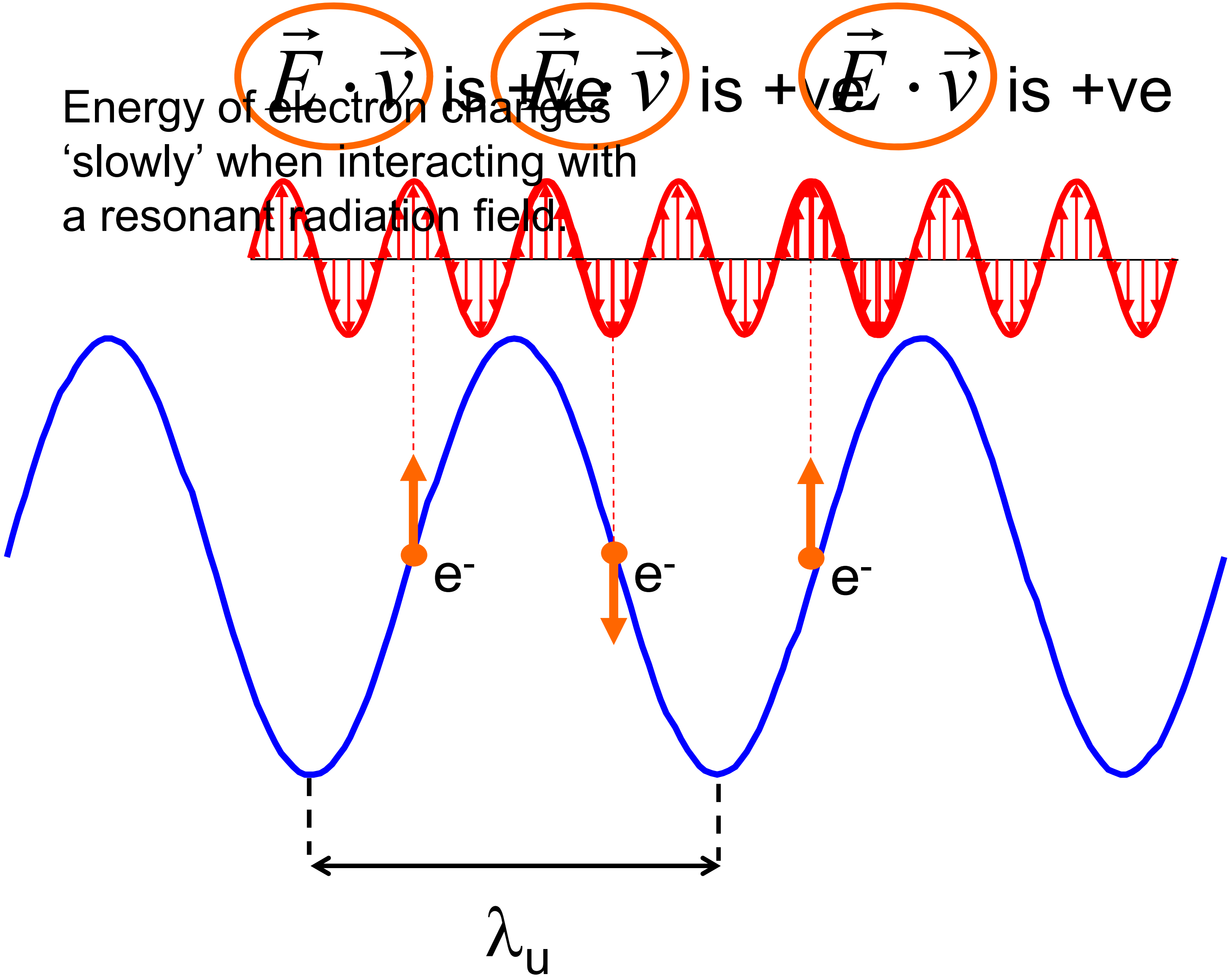
The second sin term on RHS is a wave with speed in z direction of:

$$v_z = \frac{\omega_r}{k_r - k_u} = \frac{ck_r}{k_r - k_u} \quad \Rightarrow \quad \beta_z = \frac{k_r}{k_r - k_u} > 1$$

– fast, non-resonant, phase variation.

Resonant emission – electron energy change

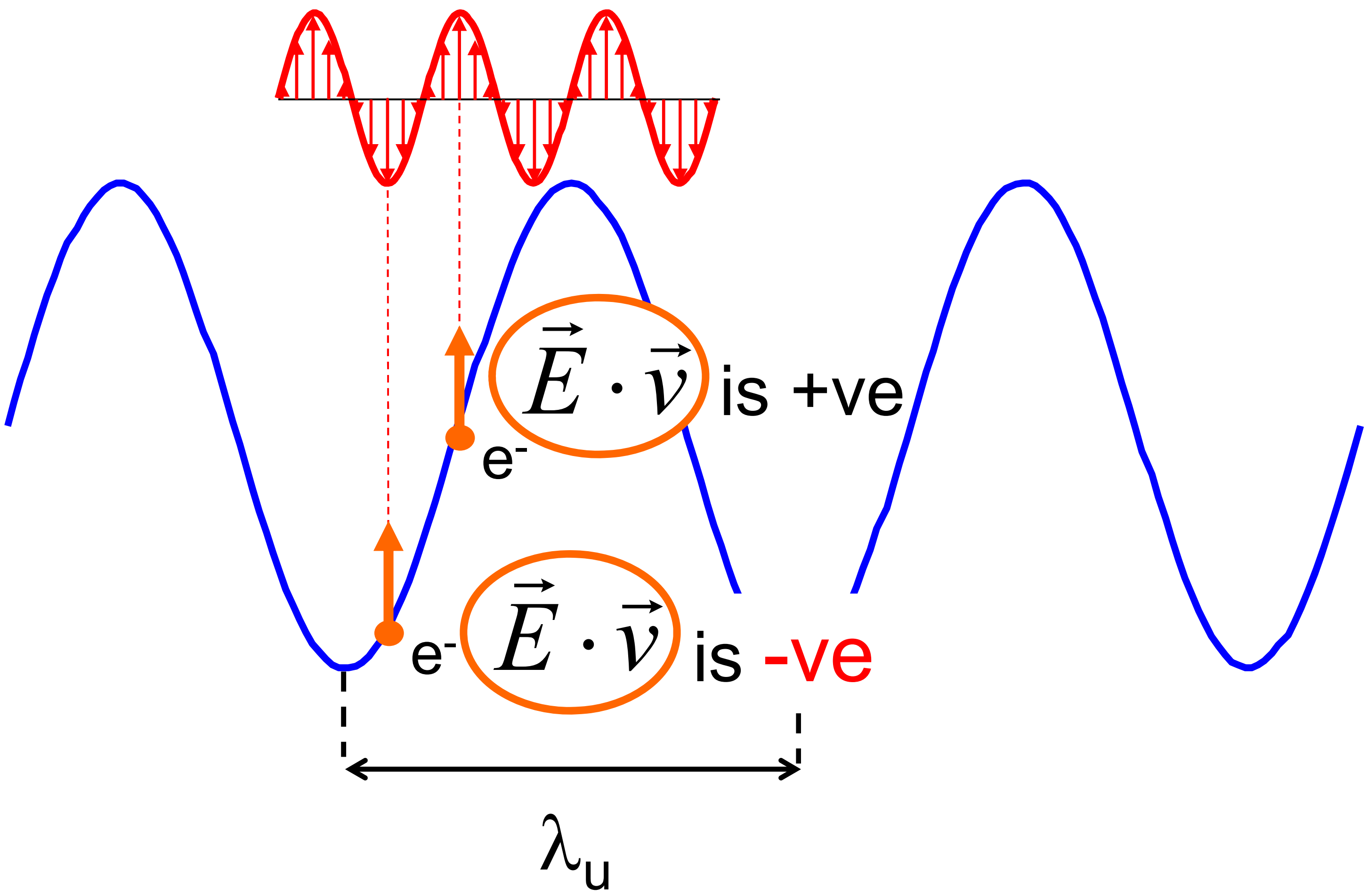
Energy of electron changes ‘slowly’ when interacting with a resonant radiation field.



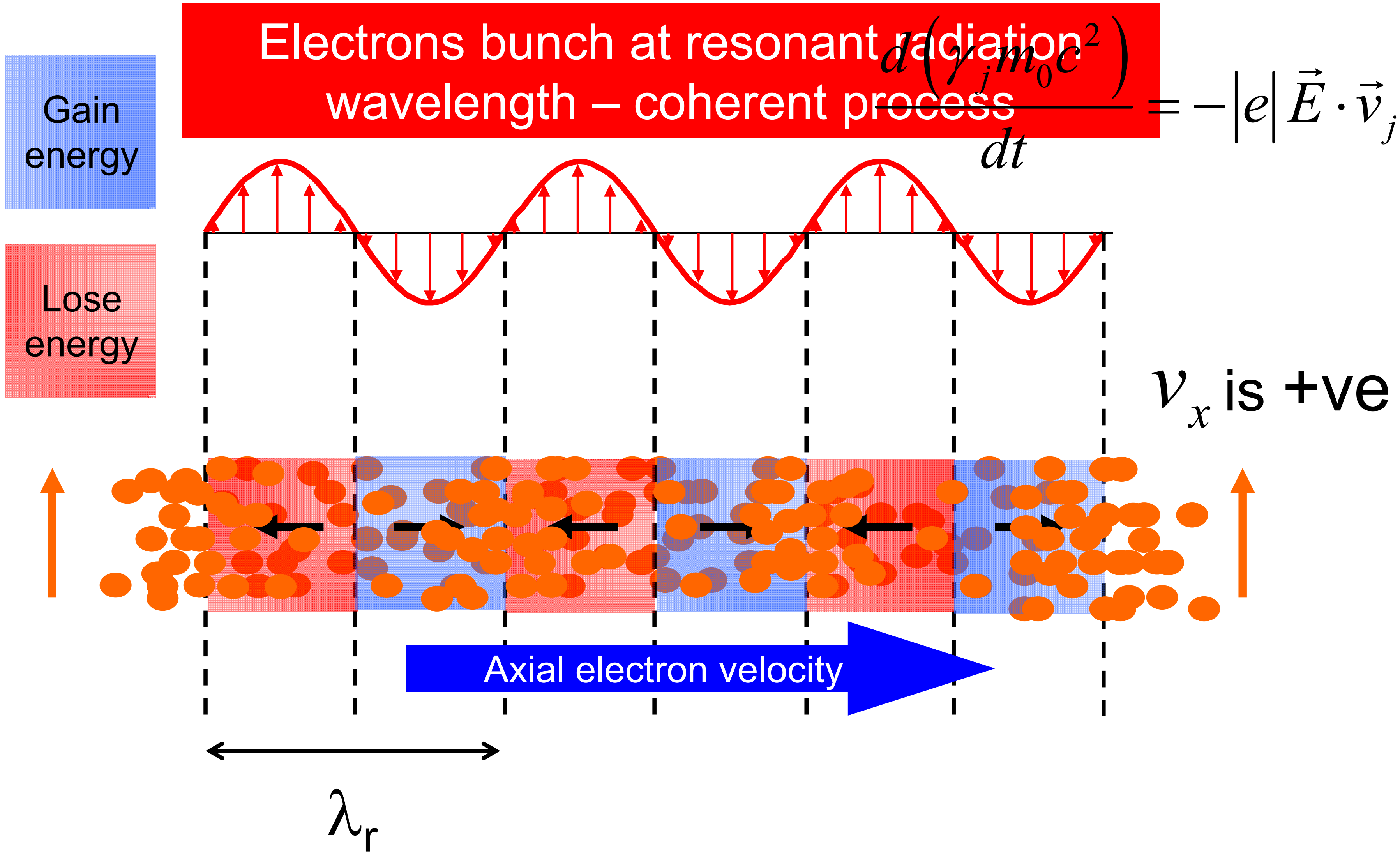
Resonant emission – electron energy change

Rate of electron energy change is 'slow' but changes periodically with respect to the radiation phase

For an electron with a different phase with respect to radiation field:

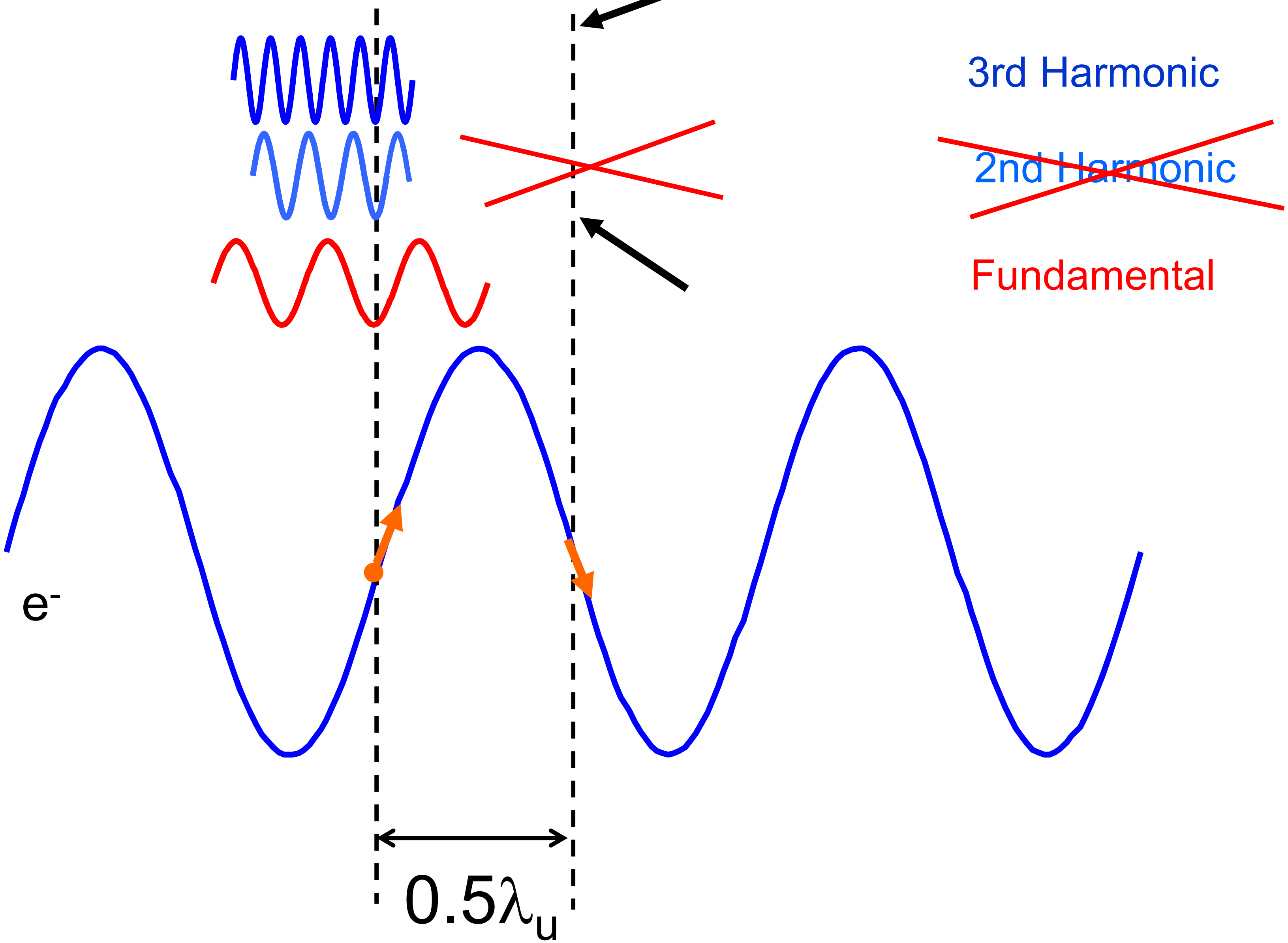


Resonant emission – electron bunching



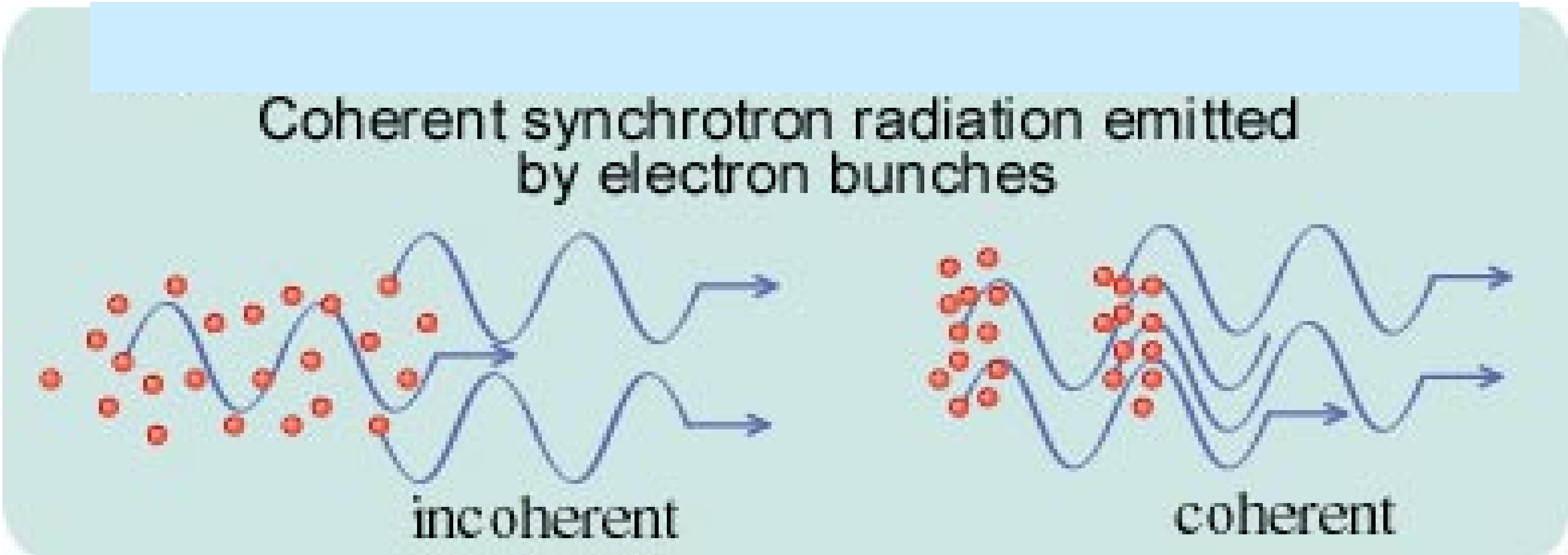
$$\frac{d\left(\gamma_j m_0 c^2\right)}{dt} = -|e|\vec{E}\cdot\vec{v}_j$$

Resonant emission - constructive interference



Even harmonics do not allow a slow exchange of energy

Bunched electrons can exchange energy coherently with radiation



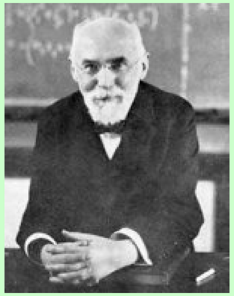
$$\text{Radiation power} \propto \left(\sum_{j=1}^{N} E_j \, e^{i\phi_j} \right)^2 = \sum_{j=1}^{N} E_j^2 + \sum_{\substack{j=1 \\ j \neq k}}^{N} \sum_{k=1}^{N} E_j E_k^* \, e^{i(\phi_j - \phi_k)}$$

If $\phi_j \approx \phi_k \;\forall\, j$ then the 2nd term >> 1st term as there are N^2 of them and results in coherent emission.

Electron bunching in a self-consistent radiation field

Basic FEL mechanism

Radiation field bunches electrons



$$\underset{\sim}{F}_j = -|e|\left[\underset{\sim}{E} + \underset{\sim}{v}_j \times \underset{\sim}{B}\right]$$

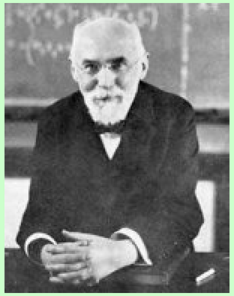
Bunched electrons drive radiation

$$\nabla^2 \underset{\sim}{E} - \frac{1}{c^2}\frac{\partial^2 \underset{\sim}{E}}{\partial t^2} = \mu_0 \frac{\partial \underset{\sim}{J}_\perp}{\partial t}$$

$$\underset{\sim}{J}_\perp \equiv -|e| \sum_{j=1}^{N} \underset{\sim}{v}_\perp \, \delta\left(\underset{\sim}{r} - \underset{\sim}{r}_j(t)\right)$$ The transverse current density

Basic FEL mechanism

Radiation field bunches electrons



$$\frac{d\theta_j}{d\bar{z}} = p_j$$

$$\frac{dp_j}{d\bar{z}} = -\left(Ae^{i\theta_j} + c.c.\right)$$

$A(\bar{z},\bar{t}) \propto E(\bar{z},\bar{t})$ ~~$\sin(k_r z - \omega_r t + \phi)$~~ – Radiation envelope

Bunched electrons drive radiation

$$\frac{\partial A}{\partial \bar{z}} + \frac{\partial A}{\partial \bar{t}} = b(\bar{z},\bar{t})$$

$$b(\bar{z},\bar{t}) = \frac{I(\bar{t})}{I_{pk}}\left(\frac{1}{N}\sum_{j=1}^{N} e^{-i\theta_j(\bar{z})}\right)\Bigg|_{\bar{t}}$$

$$\theta_j \equiv (k_r + k_u) z_j - \omega_r t$$

$$p_j \equiv \frac{\gamma_j - \gamma_r}{\rho \gamma_r}$$

$$\rho |A|^2 \equiv \frac{P_{rad}}{P_{beam}}$$

$$\rho = \frac{1}{\gamma_r}\left(\frac{\bar{a}_w \omega_p f_B}{4ck_w}\right)^{2/3}.$$

$$\omega_p = (e^2 n_{pk}/\epsilon_0 m)^{1/2}$$

$$f_B = J_0(\zeta) - J_1(\zeta)$$

$$\zeta = \bar{a}_w^2/2(1+\bar{a}_w^2)$$

$$\bar{z} = \frac{z}{l_g}, \quad \bar{t} = \frac{z - c\bar{\beta}_z t}{l_c}$$

$$l_g = \lambda_w/4\pi\rho$$

$$l_c = \lambda_r/4\pi\rho$$

These equations are assumed 'slowly varying' i.e. any evolution is assumed slow with respect to the radiation/undulator period. They can be subsequently averaged over a radiation/undulator period.

Conventional laser Vs FEL pulses

Linear analysis

$$\frac{d\theta_j}{d\bar{z}} = p_j$$

$$\frac{dp_j}{d\bar{z}} = -\left(Ae^{i\theta_j} + c.c.\right)$$

$$\frac{\partial A}{\partial \bar{z}} + \cancel{\frac{\partial A}{\partial \bar{t}}} = b\left(\bar{z}, \cancel{\bar{t}}\right)$$

Steady-state approx.: “No pulses”

Assume that:

$$A_0 \ll 1$$

$$p_{j_0} = \delta$$

$$\theta_j = \theta_{0j} + \theta_{1j} \quad \text{etc.} \qquad \text{where: } \theta_{1j} \ll 1$$

$$\left\langle e^{-i\theta_0} \right\rangle = 0; \qquad \theta_{0j} = U\left(0, 2\pi\right]$$

Using: $e^x \approx 1 + x + \dfrac{x^2}{2} + \ldots$

$$\Rightarrow e^{i\left(\theta_{0j} + \theta_{1j}\right)} = e^{i\theta_{0j}} e^{i\theta_{1j}} \approx e^{i\theta_{0j}} \left(1 + i\theta_{1j}\right)$$

$$\left\langle x \right\rangle \equiv \frac{1}{N} \sum_{j=1}^{N} x_j$$

$$\Rightarrow \quad \frac{db}{d\bar{z}} = -iP - i\delta b$$

$$\frac{dP}{d\bar{z}} = -A + i\delta P$$

$$\frac{dA}{d\bar{z}} = b$$

Where:

$$b \equiv \left\langle -i\theta_1 e^{-i\theta_0} \right\rangle$$

$$P \equiv \left\langle p_1 e^{-i\theta_0} \right\rangle$$

The steady-state approximation can be thought of as the continuous e⁻ beam limit where the electron ‘pulse’ has no beginning or end. In this case one can see that the radiation field can only be a function of the distance through the undulator and no pulse effects can be present.

Linear analysis

First assume resonance: $\delta = 0$

Differentiating linear equations:

$$\frac{d^2A}{d\bar{z}^2} = \frac{db}{d\bar{z}} = -iP$$

$$\Rightarrow \frac{d^3A}{d\bar{z}^3} = -i\frac{dP}{d\bar{z}} = iA$$

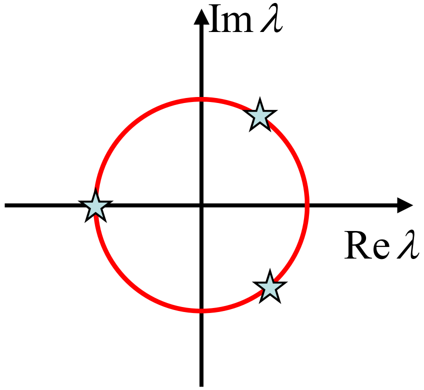
Look for solutions:

$$A(\bar{z}) = A_0 e^{i\lambda \bar{z}}$$

$$\Rightarrow -i\lambda^3 = i$$

$$\Rightarrow \lambda^3 = -1$$

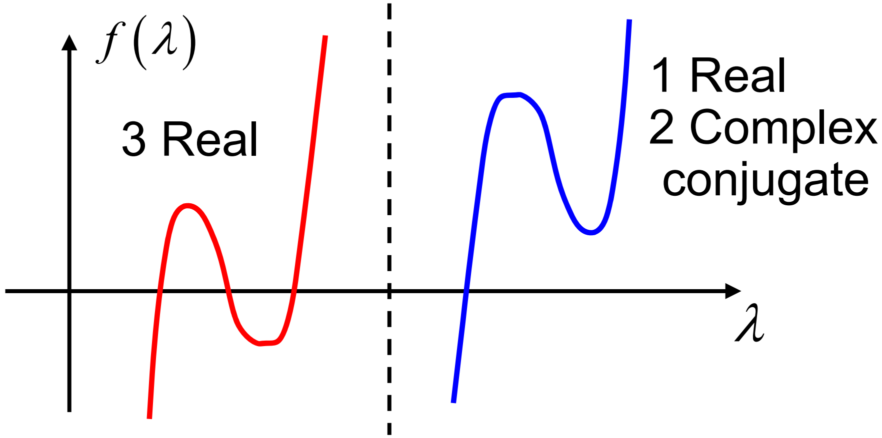
$$\Rightarrow \lambda = -1; \left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right); \left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$$



Away from resonance: $\delta \neq 0$

the dispersion relation is:

$$f(\lambda) = \lambda^3 - \delta\lambda^2 + 1 = 0$$



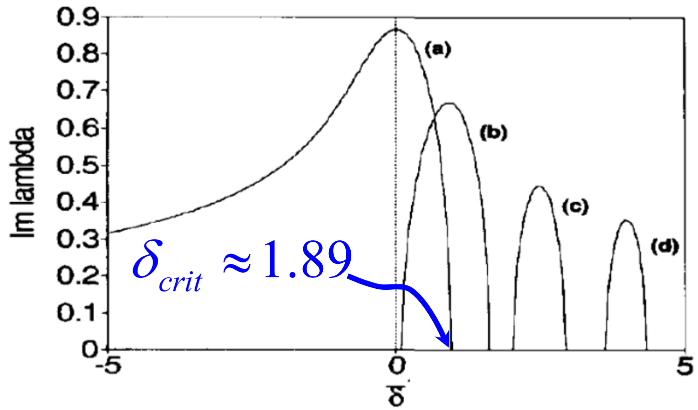


Fig. 3. Im λ as a function of δ for (a) $\sigma_\epsilon = 0$; (b) $\sigma_\epsilon = 2.0$; (c) $\sigma_\epsilon = 5.0$; (d) $\sigma_\epsilon = 8.0$.

Linear analysis

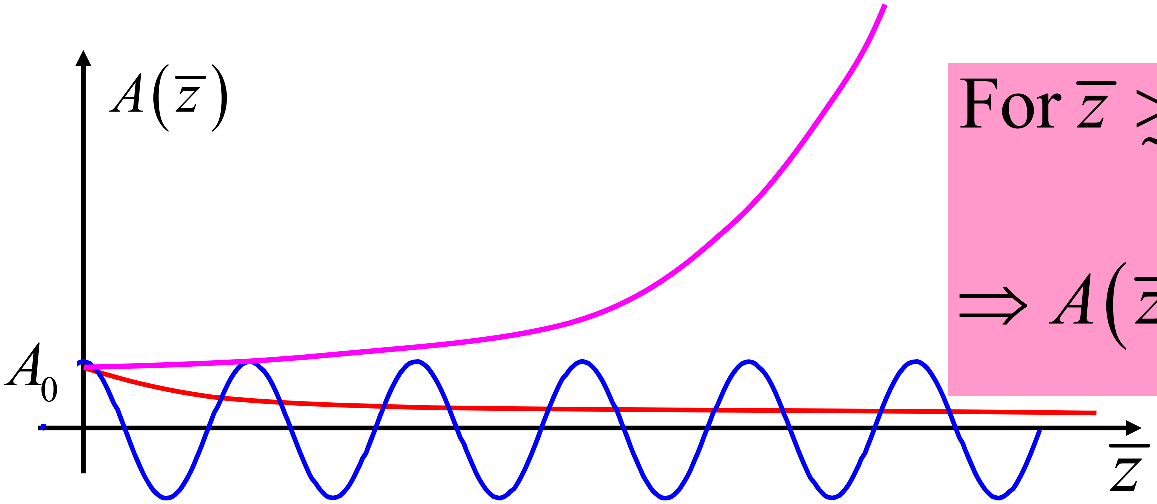
Solutions for $\delta = 0$:

$$A(\bar{z}) = \frac{A_0}{3} \sum_j c_j e^{i\lambda_j \bar{z}} \quad \text{for} \quad \lambda_j = \left[-1;\ \left(\frac{1}{2} + i\frac{\sqrt{3}}{2} \right);\ \left(\frac{1}{2} - i\frac{\sqrt{3}}{2} \right) \right]$$

Real parts give oscillatory solutions.

Imaginary parts give exponential growth: $\left(-i\frac{\sqrt{3}}{2} \right)$

and exponential decay: $\left(+i\frac{\sqrt{3}}{2} \right)$



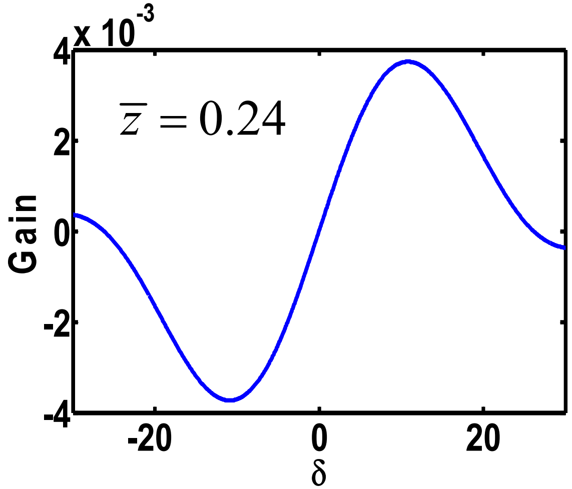
For $\bar{z} \gtrsim 1$

$$l_g = \lambda_w / 4\pi\rho$$

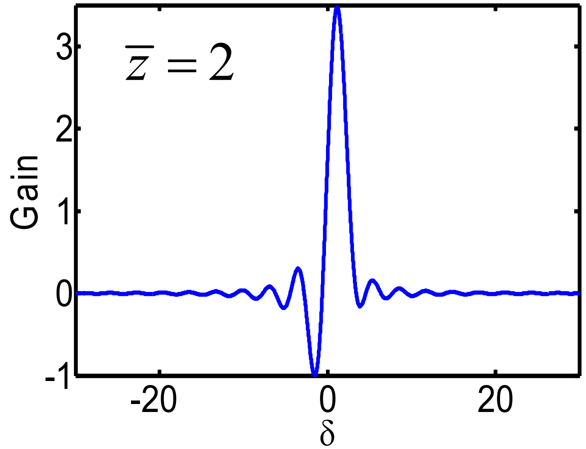
$$\Rightarrow A(\bar{z}) \approx \frac{A_0}{3} e^{\frac{\sqrt{3}}{2}\bar{z}} = \frac{A_0}{3} e^{\frac{\sqrt{3}}{2}\frac{z}{l_g}}$$

Gain as a function of detuning from resonance

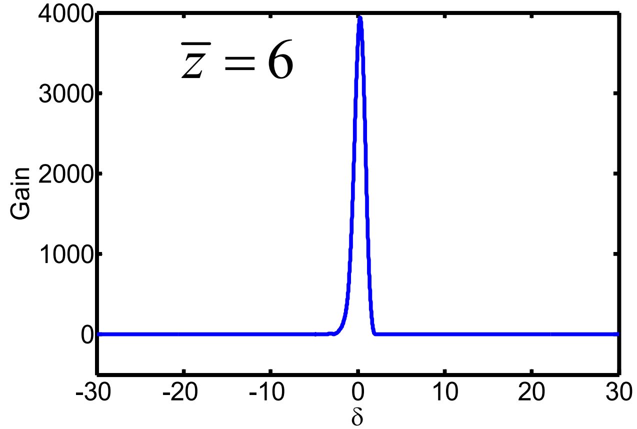
$$\text{Gain} = \frac{\left|A(\bar{z})\right|^2 - \left|A_0\right|^2}{\left|A_0\right|^2}$$



Oscillatory terms dominate



Oscillatory & exponential



+ve exponential term dominates

$$\delta \equiv p_{j0} = \frac{\gamma_{j0} - \gamma_r}{\rho \gamma_r}$$

Constants of motion

$$\frac{d\theta_j}{d\bar{z}} = p_j$$

$$\frac{dp_j}{d\bar{z}} = -\left(Ae^{i\theta_j} + c.c.\right)$$

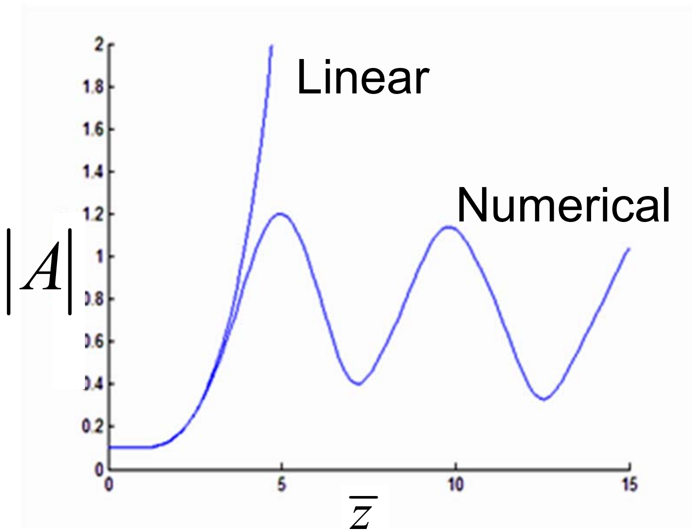
$$\frac{dA}{d\bar{z}} = b(\bar{z}, \bar{t})$$

Two constants of motion can be obtained from these equations in the steady-state limit:

$$|A|^2 + \langle p \rangle = \text{ constant}$$

$$\frac{\langle p^2 \rangle}{2} + i\left(A^* b - Ab^*\right) - \delta |A|^2 = \text{ constant}$$

Where the constant is the variables' initial values.

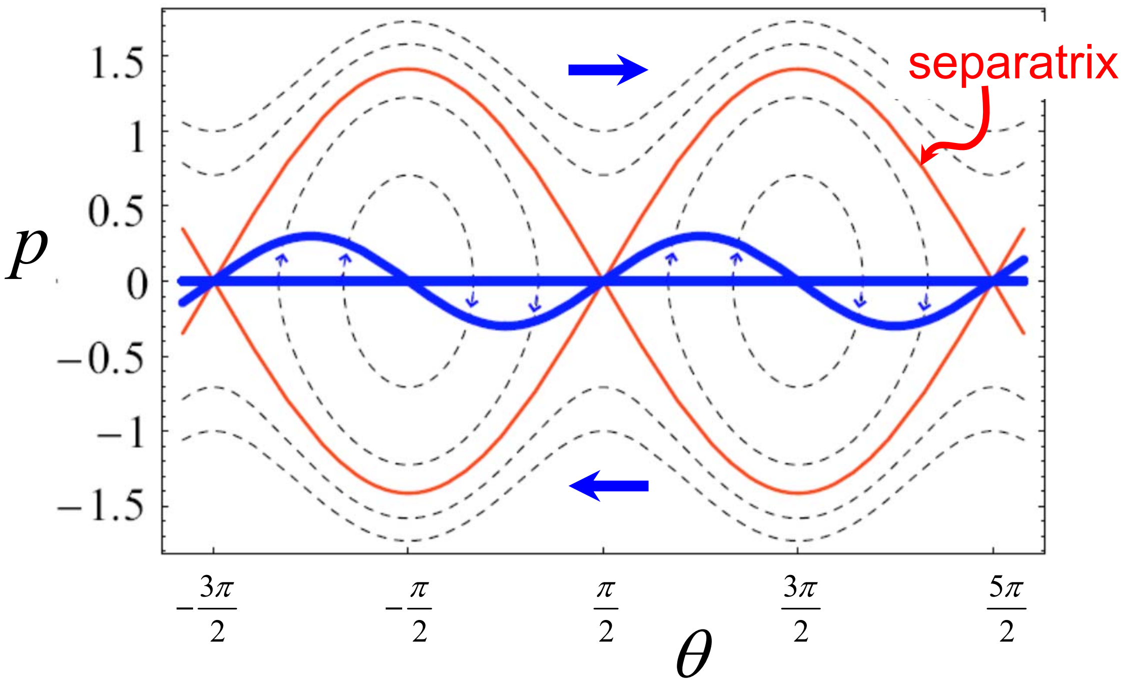


The first constant above corresponds to conservation of energy. The second, incorporating phase dependent terms is related to the Hamiltonian of the system. Opposite is plotted the linear and non-linear (numerical) solutions of the equations for a resonant interaction $(\delta = 0)$. From the definition of :

$$\rho |A|^2 \equiv \frac{P_{rad}}{P_{beam}}$$

and the saturated scaled field $|A_{sat}|$~1, it is seen that ρ is a measure of the efficiency if the interaction.

The pendulum equation and phase-space



$$\frac{d\theta_j}{d\bar{z}} = p_j$$

$$\frac{dp_j}{d\bar{z}} = -2a\cos\left(\theta_j + \phi\right)$$

$$\frac{da}{d\bar{z}} = \left\langle \cos\left(\theta + \phi\right) \right\rangle$$

$$\frac{d\phi}{d\bar{z}} = -\frac{1}{a}\left\langle \sin\left(\theta + \phi\right) \right\rangle$$

The electrons can be thought of as a collection of pendula initially distributed over a range of angles with respect to the vertical. The radiation field is analogous to the gravitational field. The separartrix defines the boundary between pendula that librate and rotate. Of course in the FEL equations above, unlike a gravitational field, the radiation field can evolve in both amplitude a, and phase ϕ.