Nonlinear Single-Particle Dynamics in High Energy Accelerators

Part 2

Second Example: Nonlinear dynamics in storage rings

Nonlinear Single-Particle Dynamics in High Energy Accelerators

There are six lectures in this course on nonlinear dynamics:

1. First example: nonlinear dynamics in a bunch compressor
2. Second example: nonlinear dynamics in storage rings
3. Hamiltonian mechanics
4. Canonical perturbation theory
5. Lie transformations
6. Symplectic integrators

As a second example of nonlinear dynamics in accelerator systems, let us consider the transverse dynamics in a simple storage ring.

We shall assume that:

- The storage ring is constructed from some number of identical cells consisting of dipoles, quadrupoles and sextupoles.
- The phase advance per cell can be tuned from close to zero, up to about $0.5\times 2\pi$.
- There is one sextupole per cell, which is located at a point where the horizontal beta function is 1 m, and the alpha function is zero.

Usually, a storage ring will contain two sextupoles per cell, to correct horizontal and vertical chromaticity. To keep things simple, we will use only one sextupole per cell.

Storage ring: linear dynamics

The chromaticity, and hence the sextupole strength, will normally be a function of the phase advance. However, just to investigate the system, let us keep a fixed sextupole strength $k_2 L$, and see what happens as we adjust the phase advance.

We can assume that the map from one sextupole to the next is linear, and corresponds to a rotation in phase space through an angle given by the phase advance μ_x.

With the sextupole strength set to zero, the transfer map for a single cell can be written:

$$\begin{pmatrix} x \\ p_x \end{pmatrix}_{s_0+L_{\text{cell}}} = \begin{pmatrix} \cos\mu_x & \sin\mu_x \\ -\sin\mu_x & \cos\mu_x \end{pmatrix} \begin{pmatrix} x \\ p_x \end{pmatrix}_{s_0}. \qquad (1)$$

To keep things simple, we shall consider only horizontal motion, and assume that the vertical coordinate $y = 0$ throughout.

The change in the horizontal momentum of a particle moving through the sextupole is found by integrating the Lorentz force:

$$\Delta p_x = -\int_0^L \frac{B_y}{B\rho}\,ds. \quad (2)$$

The sextupole strength k_2 is defined by:

$$k_2 = \frac{1}{B\rho}\frac{\partial^2 B_y}{\partial x^2}, \quad (3)$$

where $B\rho$ is the beam rigidity. For a pure sextupole field (assuming that the vertical coordinate $y = 0$),

$$\frac{B_y}{B\rho} = \frac{1}{2}k_2 x^2. \quad (4)$$

If the sextupole is short, then we can neglect the small change in the coordinate x as the particle moves through the sextupole, in which case:

$$\Delta p_x \approx -\frac{1}{2}k_2 L x^2. \quad (5)$$

The transfer map for a particle moving through a short sextupole can be represented by a "kick" in the horizontal momentum:

$$x_1 = x_0 \quad (6)$$

$$p_{x1} = p_{x0} - \frac{1}{2}k_2 L x_0^2. \quad (7)$$

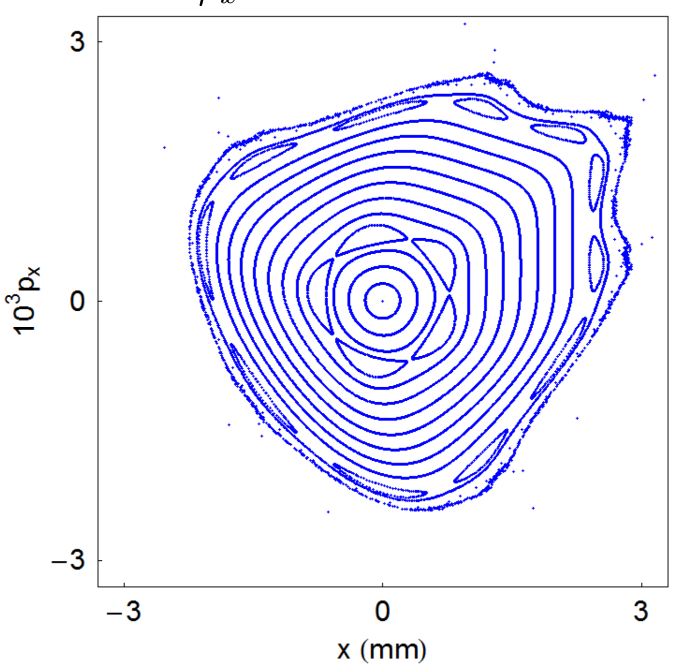
We shall choose a fixed value $k_2 L = -600\,\mathrm{m}^{-3}$, and look at the effect of the maps for different phase advances.

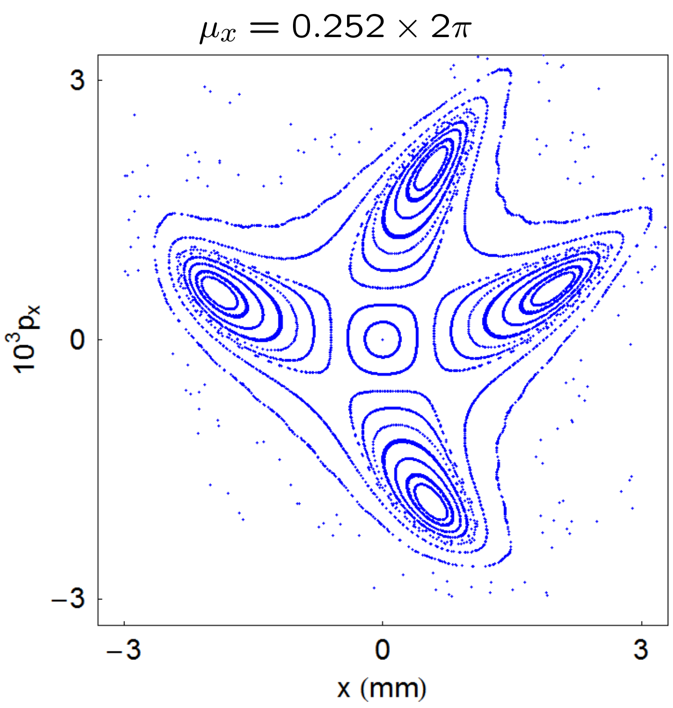
We examine the effect of the map in a given case by plotting the phase space co-ordinates after repeated action of the map (equation (1), followed by (6) and (7)) for a range of initial conditions.

The resulting plot is known as a *phase space portrait*.

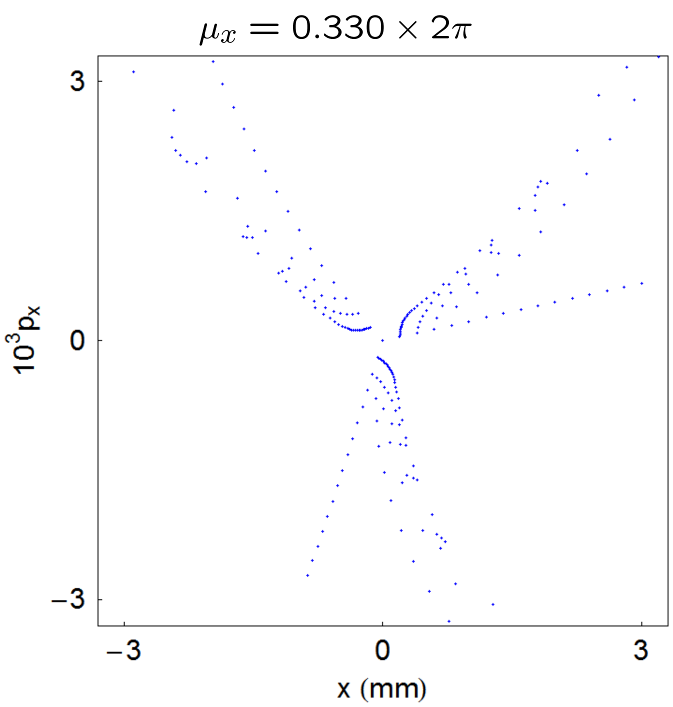
We look first at the phase space portraits for a range of phase advances from $0.2 \times 2\pi$ to $0.5 \times 2\pi$...

$\mu_x = 0.202 \times 2\pi$

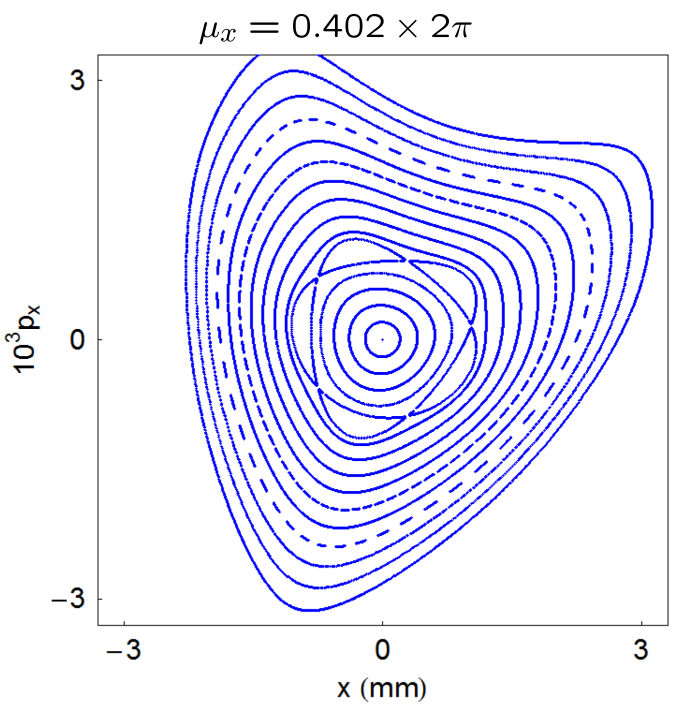




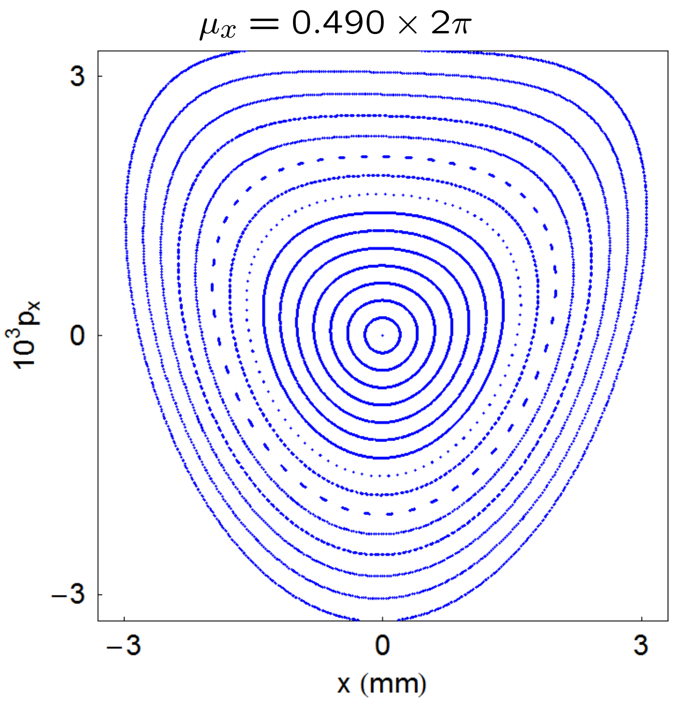
$\mu_x = 0.252 \times 2\pi$
3
0
−3
$10^3 p_x$
−3
0
3
x (mm)



$\mu_x = 0.330 \times 2\pi$
3
0
−3
$10^3 p_x$
−3
0
3
x (mm)



Phase space portraits for a storage ring with sextupoles



There are some interesting features in these phase space portraits to which it is worth drawing attention:

- For small amplitudes (small x and p_x), particles map out closed loops around the origin: this is what we expect for a purely linear map.
- As the amplitude is increased, there appear "islands" in phase space: the phase advance (for the linear map) is generally close to one divided by the number of islands.
- Sometimes, a larger number of islands appears at larger amplitude.
- Usually, there is a closed curve that divides a region of stable motion from a region of unstable motion. Outside that curve, the amplitude of particles increases without limit as the map is repeatedly applied.
- The area of the stable region depends strongly on the phase advance: for a phase advance close to $2\pi/3$, it appears that the stable region almost vanishes altogether.
- It appears that as the phase advance is increased towards π, the stable area becomes large, and distortions from the linear ellipse become less evident.

We see already that the effect of the sextupole in the lattice depends strongly on the (linear) phase advance across a single periodic cell.

In the language of beam dynamics, a phase advance of $2\pi m/n$, where m/n is an irreducible fraction, is said to be an "n^{th} order resonance".

Much of the rest of this course will be devoted to developing an understanding of the various phenomena that we have observed in this example, including resonances.

As a first step, we can consider some simple special cases...

Consider what happens if, instead of the sextupole, we have a small dipole (steering) error (of strength $k_0 L$) at one point in each cell.

The transfer map representing the dipole error is:

$$x_1 = x_0, \tag{8}$$
$$p_{x1} = p_{x0} - k_0 L. \tag{9}$$

Suppose that the phase advance from a given point in one cell to the corresponding point in the next cell is exactly 2π.

After applying the transfer map for the dipole error and then the map for one periodic cell, the phase space co-ordinates become:

$$x_2 = x_1 = x_0, \tag{10}$$
$$p_{x2} = p_{x1} = p_{x0} - k_0 L. \tag{11}$$

Then applying the next steering error:

$$x_3 = x_2 = x_0, \tag{12}$$
$$p_{x3} = p_{x2} - k_0 L = p_{x0} - 2k_0 L. \tag{13}$$

The steering errors add coherently. As a result, the motion of particles in the ring will be unstable.

Now suppose that the phase advance from a given point in one cell to the corresponding point in the next cell is exactly π.

After applying the transfer map for the dipole error and then the map for one periodic cell, the phase space co-ordinates become:

$$x_2 = -x_1 = -x_0, \qquad (14)$$

$$p_{x2} = -p_{x1} = -p_{x0} + k_0 L. \qquad (15)$$

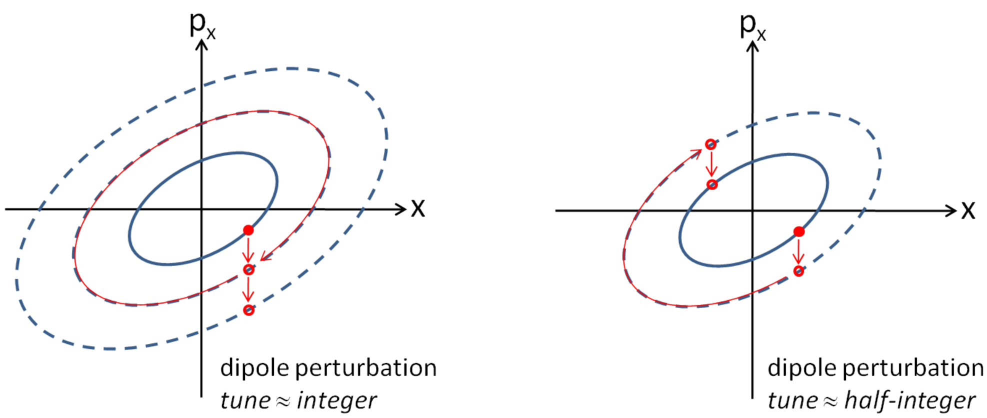
Then applying the next dipole error:

$$x_3 = x_2 = -x_0, \qquad (16)$$

$$p_{x3} = p_{x2} - k_0 L = -p_{x0}. \qquad (17)$$

The dipole errors cancel out exactly. As a result, the motion of particles in the ring will be stable.

We can understand how the effect of a dipole error depends on the phase advance by representing the effect of successive dipole kicks on a particle in phase space:



To minimise the effect of steering errors in a storage ring, it is best to choose values for the tunes close to half-integers...

...but unfortunately, a half-integer is the worst value for the tune in the presence of focusing errors.

Consider a periodic lattice in which there is a quadrupole (focusing) error at a given point in each cell.

The transfer map for the focusing error is:

$$x_1 = x_0, \tag{18}$$

$$p_{x1} = p_{x0} - k_1 L x_0. \tag{19}$$

Now suppose that the phase advance from a given point in one cell to the corresponding point in the next cell is exactly π (i.e. the tune is a half-integer).

After applying the transfer map for a quadrupole error and then the map for one periodic cell, the phase space co-ordinates become:

$$x_2 = -x_1 = -x_0, \tag{20}$$

$$p_{x2} = -p_{x1} = -p_{x0} + k_1 L x_0. \tag{21}$$

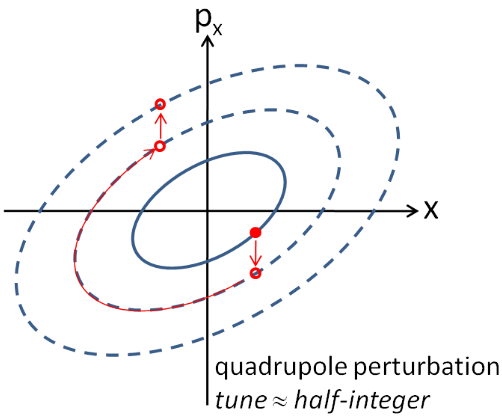
Then applying the next quadrupole error:

$$x_3 = x_2 = -x_0, \tag{22}$$

$$p_{x3} = p_{x2} - k_1 L x_2 = -p_{x0} + 2k_1 L x_0. \tag{23}$$

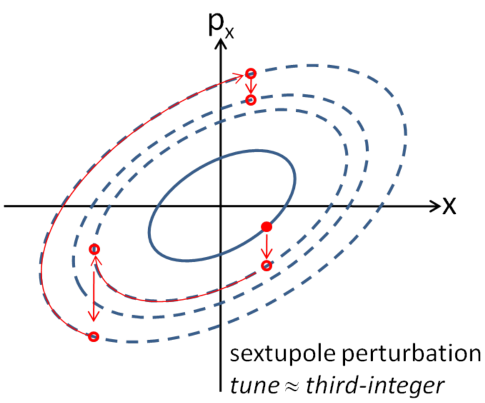
The quadrupole errors add coherently. As a result, the motion of particles in the ring will be unstable.

Again, we can understand how the effect of a quadrupole error depends on the phase advance by representing the effect of successive quadrupole kicks on a particle in phase space:



To minimise the effect of steering errors in a storage ring we need to avoid tunes close to integer values; and to minimise the effect of focusing errors, we need to avoid tunes close to half-integer values.

In the case of a sextupole perturbation, it is less clear from a simple picture of the kicks in phase space what the effects of the perturbation will be:



We might guess that the dynamics will be unstable if the (horizontal) tune is close to a third-integer resonance. Can we show this mathematically?

For analysis of sextupole (and higher-order multipole) effects, it is convenient to use action–angle variables, defined by:

$$2J_x = \gamma_x x^2 + 2\alpha_x x p_x + \beta_x p_x^2, \tag{24}$$

$$\tan(\phi_x) = -\beta_x \frac{p_x}{x} - \alpha_x, \tag{25}$$

where α_x, β_x and γ_x are the Courant–Snyder parameters.

A particle performing linear motion in an accelerator traces out an ellipse in phase space with area πJ_x.

The position around the ellipse is given by the angle variable ϕ_x.

In terms of the action J_x and angle ϕ_x, the co-ordinate and transverse momentum of a particle are given by:

$$x = \sqrt{2\beta_x J_x}\cos(\phi_x), \tag{26}$$

$$p_x = -\sqrt{\frac{2J_x}{\beta_x}}\Big[\sin(\phi_x) + \alpha_x \cos(\phi_x)\Big]. \tag{27}$$

When a particle passes through a sextupole, it receives a momentum kick:

$$\Delta p_x = -\frac{1}{2}k_2 L x^2. \tag{28}$$

Assuming that α_x is small at the sextupole location, we find that the corresponding change in the action is:

$$\Delta J_x \approx -\frac{1}{2}k_2 L\beta_x x^2 p_x = \frac{1}{8}k_2 L(2\beta_x J_x)^{\frac{3}{2}}\Big[\sin(\phi_x) + \sin(3\phi_x)\Big]. \tag{29}$$

The change in ϕ_x from a location s_0 in a beamline to a location s_1 is simply equal to the phase advance from s_0 to s_1.

We see that if each periodic cell has the same sextupole perturbation, then the sextupole kicks will add coherently if the cell tune is an integer or a third-integer.

The arguments used for a sextupole generalise to higher-order multipoles.

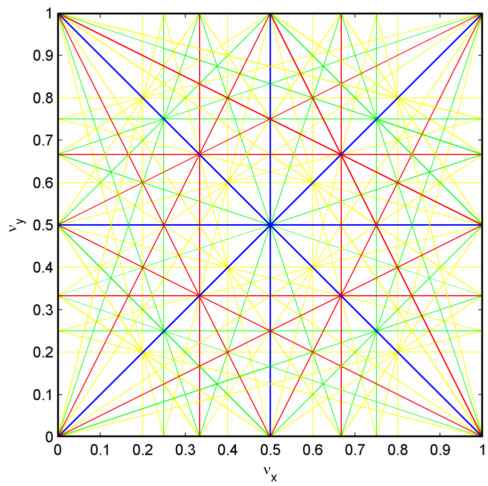
We find that resonances can occur, driven by individual multipoles or combinations of multipoles, when the tunes ν_x and ν_y for a periodic lattice satisfy:

$$m_x \nu_x + m_y \nu_y = n, \tag{30}$$

for integers m_x, m_y and n.

The resonance condition (30) can be represented by sets of lines in tune space...

The resonance diagram



The value of $|m_x| + |m_y|$ gives the order of the resonance.

Any point in tune space will be near a resonance of some order. However, it is usually the case that the higher the order of the resonance, the less dangerous the resonance tends to be for the stability of the dynamics.

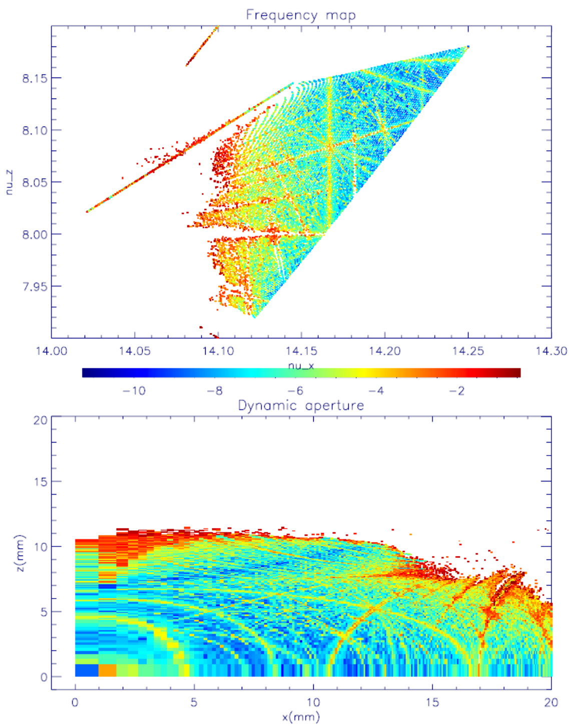
It is usually the case that particles with small betatron amplitudes (i.e. small values of the horizontal action J_x and vertical action J_y) will have stable trajectories in a storage ring; but if the betatron amplitude is larger than some limit, then the trajectory will become unstable.

The range of co-ordinates corresponding to stable trajectories in a storage ring is often called the *dynamic aperture*.

The dynamic aperture is an important quantity for injection and beam lifetime: if the dynamic aperture is very small (because of a poor choice of tunes, or because of large perturbations in the lattice) then injection efficiency will be very poor, and the beam lifetime will be very short.

The dynamic aperture depends on the energy deviation as well as on the distribution of multipoles around the lattice.

Frequency map analysis of particle dynamics in the ALS

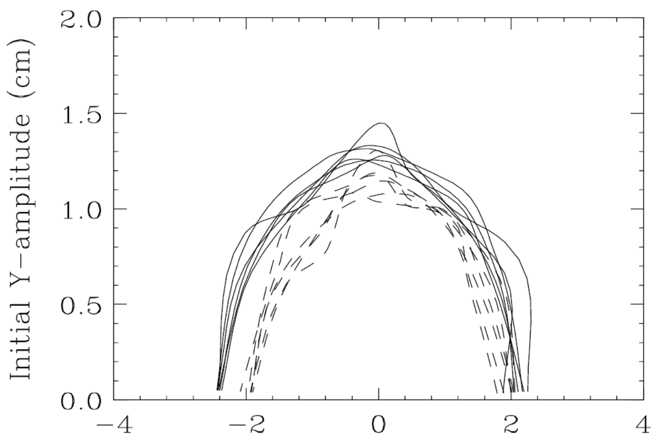


C. Steier et al., EPAC 2000, 1077.

The dynamic aperture for a given lattice design is generally computed by long-term tracking, i.e. tracking particles around the ring for thousands or tens of thousands of turns.

Particles that survive over some specified number of turns are said to be within the dynamic aperture of the lattice.

Further analysis of the tracking data, using a technique known as *frequency map analysis* can reveal the resonances that may be limiting the dynamic aperture.



J. Corbett et al., PAC 1999, 2364.

The lines show the dynamic aperture in SPEAR3 for different seeds of random multipole errors in the lattice. Solid lines are for zero energy deviation, dashed lines for 3% energy deviation.

- The nonlinear dynamics in a storage ring depend on the tunes of the storage ring, as well as on the distribution of higher-order multipoles in the lattice.
- A particular working point in tune space can be good for suppressing the effects of some multipole perturbations, but can enhance the effects of others.
- Resonances occur when multipole kicks add coherently over many turns. On a resonance, the tunes satisfy the condition:

$$m_x \nu_x + m_y \nu_y = n, \tag{31}$$

  for integers m_x, m_y and n. A good working point in tune space should avoid low-order resonances.
- The range of betatron amplitudes for which particle trajectories are stable is known as the dynamic aperture.
- The dynamic aperture in a storage ring depends on the working point in tune space, on the distribution of multipole errors around the ring, and on the energy deviation.
- A large dynamic aperture is needed for good injection efficiency in a storage ring, and good beam lifetime. Optimising the dynamic aperture is an important step in the lattice design process for a storage ring.

1. Using Matlab, Scilab, or some other scientific software, write a program to construct the phase space portraits shown in slides $8-12$. How are the phase space portraits changed if the beta function is different from unity? Make a plot of the largest stable orbit amplitude as a function of linear phase advance.
2. Consider a storage ring constructed from repeated unit cells, with a transverse phase advance of π across each cell. If there is a small focusing error k_1L at the same location in each cell, where the beta function is 1 m, write down the phase space co-ordinates after passing through N cells, starting (immediately after one focusing error) with $x = x_0$ and $p_x = 0$.
3. Using the same arguments that were used for a sextupole, show that an octupole perturbation in a periodic lattice will drive resonances when the tune is equal to an integer, a half-integer, or a quarter-integer.