Nonlinear Single-Particle Dynamics in High Energy Accelerators

Part 1

First Example: Nonlinear dynamics in a bunch compressor

Nonlinear Single-Particle Dynamics in High Energy Accelerators

There are six lectures in this course on nonlinear dynamics:

1. First example: nonlinear dynamics in a bunch compressor
2. Second example: nonlinear dynamics in storage rings
3. Hamiltonian mechanics
4. Canonical perturbation theory
5. Lie transformations
6. Symplectic integrators

By the end of the course, you should be able to:

- explain the impact of nonlinear dynamics on beam behaviour in some simple accelerator systems;
- write the Hamiltonian for standard nonlinear components such as sextupole magnets, and use the Hamiltonian to derive the equations of motion for particles in those components;
- explain how canonical perturbation theory can be applied to understand features of nonlinear dynamical systems;
- explain how nonlinear transfer maps can be represented in different forms, including power series, mixed-variable generating functions, and Lie transformations;
- use the appropriate tools from Hamiltonian mechanics to construct symplectic integrators for standard nonlinear components.

H. Goldstein, "Classical Mechanics," Addison-Wesley (2nd edition, 1980).
A classic text on classical mechanics. The chapters on the Hamilton Equations of Motion and Canonical Transformations are especially relevant. A new edition is also available.

E. Forest, "Beam Dynamics: A New Attitude and Framework," Taylor and Francis (1998).
A treasure-trove, though somewhat daunting. Very relevant.

A.J. Dragt, "Lie Methods for Nonlinear Dynamics with Applications to Accelerator Physics," (2009).
http://www.physics.umd.edu/dsat/dsatliemethods.html
Encyclopaedic. Highly recommended.

A. Wolski, "Beam Dynamics in High Energy Particle Accelerators," Imperial College Press (2014).
Follows closely the material in this course, and develops many of the ideas further.

A.W. Chao, K.H. Mess, M. Tigner, F. Zimmermann (editors), "Handbook of Accelerator Physics and Engineering," World Scientific (2nd edition, 2013).
Section 2.3 (various authors) covers nonlinear dynamics.

L.E. Reichl, "A Modern Course in Statistical Physics," (1984).
Does not mention accelerators explicitly – but the section on ergodicity contains a very relevant example, and shows how widely some of the concepts of nonlinear dynamics occur in physics.

In this lecture, we shall discuss a first example of nonlinear single-particle dynamics in an accelerator system.

In particular, we shall carry out an analysis of the longitudinal dynamics in a bunch compressor.

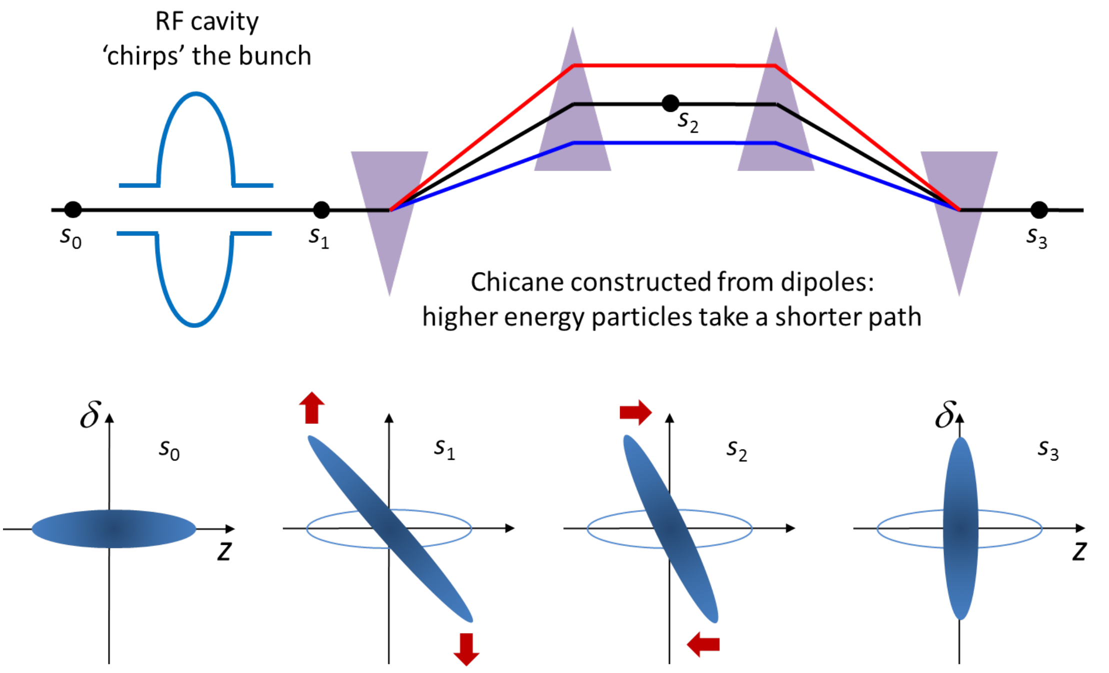
By the end of the lecture, you should be able to describe the source of nonlinearities in a bunch compressor, and the (potential) limitations imposed by the nonlinearities.

A bunch compressor is a system that reduces the length of a bunch by performing a rotation in longitudinal phase space.

Such systems are used, for example, in free electron lasers, to increase the peak current in a bunch.

We shall work through this example in some detail, almost as a case study, following these steps:

1. Outline of structure and operation of a bunch compressor.
2. Specification of parameters based on linear dynamics.
3. Analysis of linear and nonlinear effects.
4. Modification of parameters to compensate nonlinear effects.



Distribution of particles 'rotates' in longitudinal phase space (area is conserved).

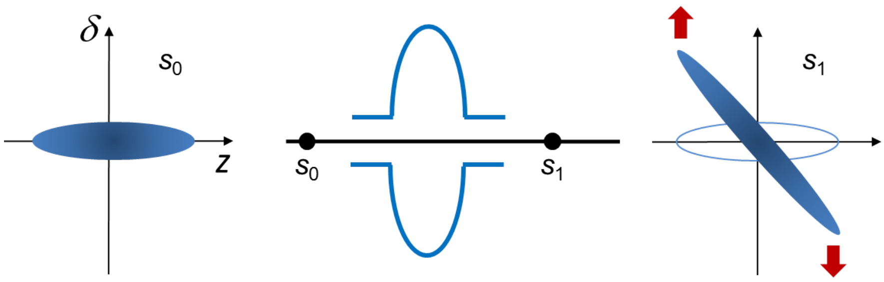
Dynamical variables

We define *dynamical variables* to describe the behaviour of individual particles as they travel along the beamline.

The energy deviation δ is the "energy error" of a particle with respect to a specified reference energy, E_0:

$$\delta = \frac{E - E_0}{E_0}. \tag{1}$$

The other dynamical variable, z, is the distance that a particle is ahead of a nominal reference particle.



The rf cavity is designed to "chirp" the bunch, i.e. to provide a change in energy deviation as a function of longitudinal position within the bunch.

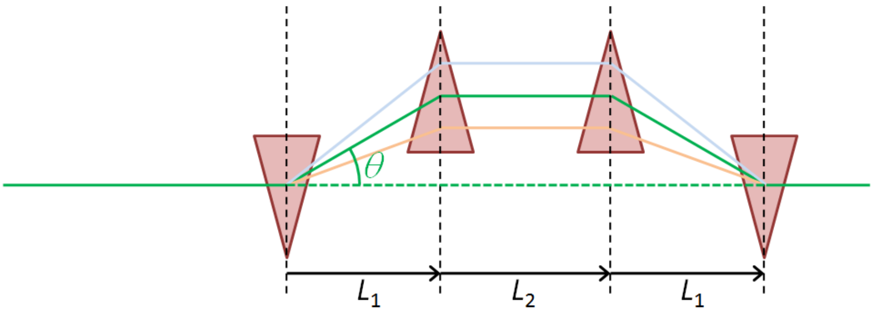
For ultrarelativistic particles, the dynamical map for the rf cavity in the bunch compressor is:

$$z_1 = z_0, \tag{2}$$

$$\delta_1 = \delta_0 - \frac{eV}{E_0}\sin\left(\frac{\omega z_0}{c}\right), \tag{3}$$

where V is the rf voltage, and ω is 2π times the rf frequency.

The chicane does not change the energy of the particles (neglecting synchrotron radiation). However, the path length L depends on the energy of the particle.



If we assume that the bending angle in a dipole is small, $\theta \ll 1$:

$$L = \frac{2L_1}{\cos\theta} + L_2. \tag{4}$$

The bending angle is a function of the energy of the particle:

$$\theta = \frac{\theta_0}{1+\delta}. \tag{5}$$

The complete map for the bunch compressor can be written as a map for the rf cavity (2), (3):

$$\begin{aligned} z_1 &= z_0, \\ \delta_1 &= \delta_0 - \frac{eV}{E_0}\sin\left(\frac{\omega z_0}{c}\right), \end{aligned}$$

followed by a map for the chicane:

$$z_3 = z_1 + 2L_1\left(\frac{1}{\cos\theta_0} - \frac{1}{\cos\theta}\right), \tag{6}$$

$$\delta_3 = \delta_1, \tag{7}$$

where θ_0 is the nominal bending angle of each dipole in the chicane, and θ is given by (5):

$$\theta = \frac{\theta_0}{1+\delta_1}.$$

Clearly, the map is nonlinear. The question is: how important are the nonlinear terms?

To understand the effect of the nonlinear part of the map, we shall look at a specific example.

First, we will "design" a bunch compressor using only the linear part of the map, i.e. by completely ignoring the nonlinear terms.

Then, we shall see how our design has to be modified to take account of the nonlinearities.

To first order in the dynamical variables z and δ, the map for the bunch compressor can be written:

$$z_1 = z_0, \tag{8}$$

$$\delta_1 = \delta_0 - \frac{eV}{E_0}\frac{\omega}{c} z_0, \tag{9}$$

followed by:

$$z_3 = z_1 + 2L_1 \frac{\theta_0 \sin\theta_0}{\cos^2\theta_0} \delta_1, \tag{10}$$

$$\delta_3 = \delta_1. \tag{11}$$

In a linear approximation, the maps for the rf cavity and the chicane may be represented as matrices:

$$R_{\mathrm{rf}} = \begin{pmatrix} 1 & 0 \\ -a & 1 \end{pmatrix}, \qquad R_{\mathrm{ch}} = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \tag{12}$$

where:

$$a = \frac{eV}{E_0}\frac{\omega}{c}, \quad \text{and} \quad b = 2L_1 \frac{\theta_0 \sin\theta_0}{\cos^2\theta_0}. \tag{13}$$

The matrix representing the total map for the bunch compressor, R_{bc}, is then:

$$R_{\mathrm{bc}} = R_{\mathrm{ch}} R_{\mathrm{rf}} = \begin{pmatrix} 1-ab & b \\ -a & 1 \end{pmatrix}. \tag{14}$$

The map is applied by multiplying the *phase space vector* by the matrix R_{bc}:

$$\begin{pmatrix} z \\ \delta \end{pmatrix}_{s=s_3} = R_{\mathrm{bc}} \begin{pmatrix} z \\ \delta \end{pmatrix}_{s=s_0}. \tag{15}$$

We note in passing that the linear part of the map is *symplectic*.

A linear map is symplectic if the matrix R representing the map is symplectic, i.e. satisfies:

$$R^{\mathsf{T}} S R = S, \tag{16}$$

where, in one degree of freedom (i.e. two dynamical variables), S is the matrix:

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{17}$$

In more degrees of freedom, S is constructed by repeating the 2×2 matrix above on the block diagonal, as often as necessary.

In one degree of freedom, it is a necessary and sufficient condition for a matrix to be symplectic, that it has unit determinant: but this condition does *not* generalise to more degrees of freedom.

We shall consider what it means to say that a nonlinear map is symplectic later in this course.

Now we proceed to derive expressions for the required values of the parameters a and b, in terms of the desired initial and final bunch length and energy spread.

We construct the beam distribution *sigma matrix* by taking the outer product of the phase space vector for each particle, then averaging over all particles in the bunch:

$$\Sigma = \langle \vec{z}\,\vec{z}^{\mathsf{T}} \rangle = \begin{pmatrix} \langle z^2 \rangle & \langle z\delta \rangle \\ \langle z\delta \rangle & \langle \delta^2 \rangle \end{pmatrix}. \tag{18}$$

The transformation of Σ under the linear map represented by the matrix R_{bc} is given by:

$$\Sigma_{s_3} = R_{\mathrm{bc}}\, \Sigma_{s_0}\, R_{\mathrm{bc}}^{\mathsf{T}}. \tag{19}$$

Usually, a bunch compressor is designed so that the correlation $\langle z\delta \rangle = 0$ at the start *and end* of the compressor.

In that case, using (14) and (19) we find that the parameters a and b must satisfy:

$$(1 - ab)\frac{a}{b} = \frac{\langle \delta^2 \rangle_{s_0}}{\langle z^2 \rangle_{s_0}} \tag{20}$$

where the subscript s_0 indicates that the average is taken over the values of the dynamical variables at $s = s_0$.

Given the constraint (20), the compression factor r is given by:

$$r^2 \equiv \frac{\langle z^2 \rangle_{s_3}}{\langle z^2 \rangle_{s_0}} = 1 - ab. \tag{21}$$

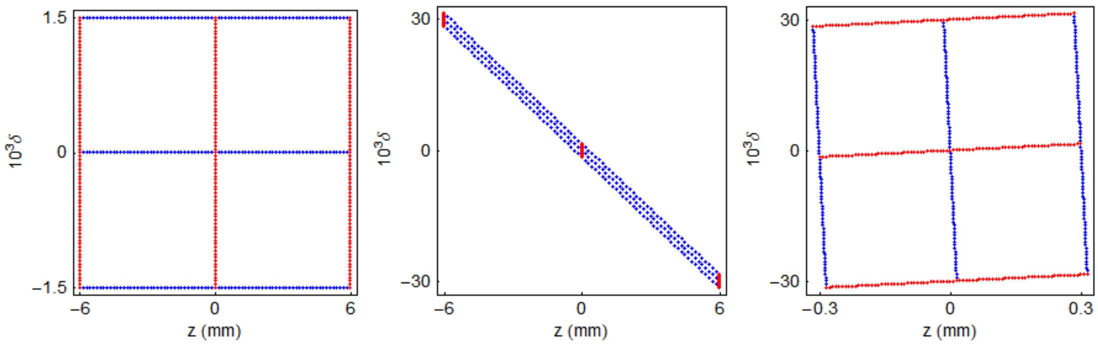
As a specific example, consider a bunch compressor for the International Linear Collider:

Initial rms bunch length	$\sqrt{\langle z^2 \rangle_{s_0}}$	6 mm
Initial rms energy spread	$\sqrt{\langle \delta^2 \rangle_{s_0}}$	1.5×10^{-3}
Final rms bunch length	$\sqrt{\langle z^2 \rangle_{s_3}}$	0.3 mm

Solving equations (20) and (21) with the above values for rms bunch lengths and energy spread, we find:

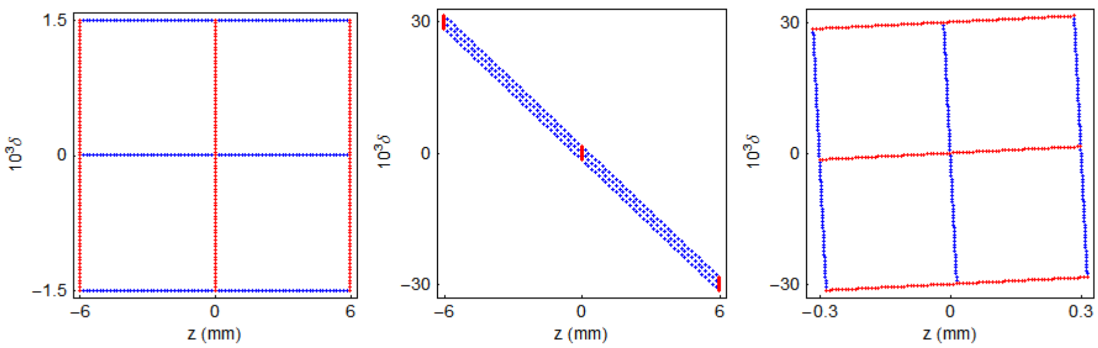
$$a = 4.9937\,\mathrm{m}^{-1}, \quad \text{and} \quad b = 0.19975\,\mathrm{m}. \tag{22}$$

We can illustrate the effect of the linearised bunch compressor map on phase space using a "window frame" distribution:



The bunch compressor rotates the distribution in phase space by (nearly) 90°.

The rms bunch length is reduced by a factor of 20; at the same time, the rms energy spread is *increased* by the same factor.



Because the map is symplectic, phase space areas are conserved under the transformation.

Also, because the map is linear, straight lines in phase space remain straight.

Now let us see what happens when we apply the full nonlinear map for the bunch compressor to a distribution of particles.

The full map cannot simply be represented by the two parameters a and b: we need to make some assumptions for other parameters, in particular for the rf voltage and frequency, and the dipole bending angle and chicane length.

We have to choose these parameters so that the "linear" parameters have the appropriate values. Fortunately, this is not difficult.

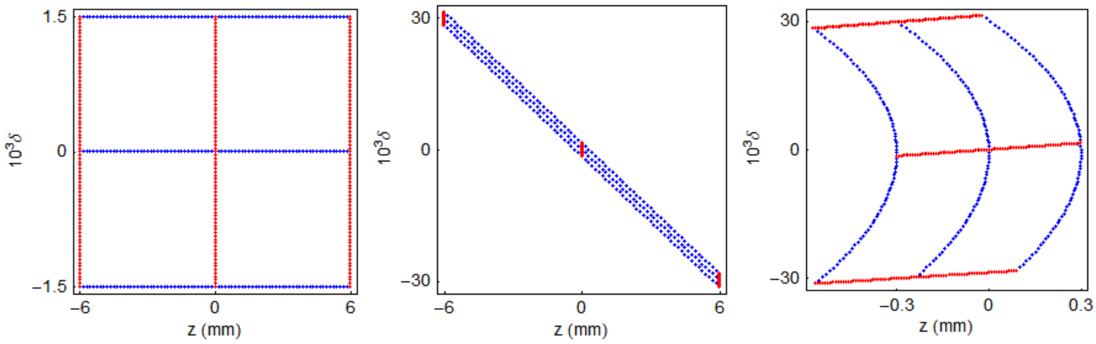
Suitable values for the various parameters are as follows:

Beam (reference) energy	E_0	5 GeV
RF frequency	f_{rf}	1.3 GHz
RF voltage	V_{rf}	916 MV
Dipole bending angle	θ_0	3°
Dipole spacing	L_1	36.3 m

It appears that we need a lot of rf voltage; the design is still feasible, though expensive.

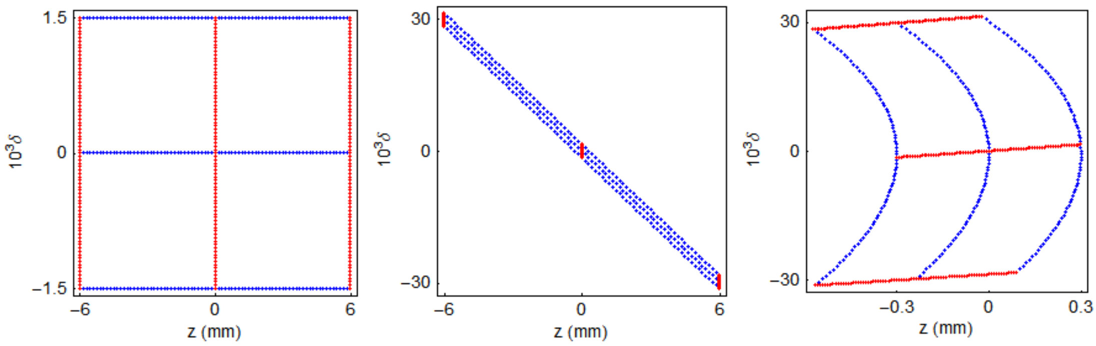
Let us see what happens to the dynamics when we use these parameters...

As before, we illustrate the effect of the bunch compressor map on phase space using a "window frame" distribution:



The map has approximately the effect we desire: the bunch length has been reduced (and the distribution rotated by approximately 90°).

However, there is significant distortion of the distribution.



Because of the nonlinear terms in the map, straight lines do not stay straight.

The rms bunch length will be significantly longer than we are aiming for.

Whether or not the nonlinear effects can be tolerated will depend on the application. In the case of ILC, the phase space distortion introduced by a bunch compressor with the above parameters would lead to a significant loss of luminosity. We have to do something about it... but what?

If we inspect the phase space plots, then it seems that the damage is done by a second-order term in the map for the chicane, i.e. by a dependence of a change in z on the square of the energy deviation δ: such a term is a possible cause of the "parabolic" disortion that we see in the final phase space plot.

Assuming that our conjecture is correct, we could try to fix the distortion by modifying the map for the rf...

Consider a particle entering the bunch compressor with initial phase space co-ordinates z_0 and δ_0.

We can write the co-ordinates z_1 and δ_1 of the particle after the rf cavity *to second order* in z_0 and δ_0:

$$z_1 = z_0, \tag{23}$$

$$\delta_1 = \delta_0 + (R_{\text{rf}})_{65} z_0 + (T_{\text{rf}})_{655} z_0^2. \tag{24}$$

$(R_{\text{rf}})_{65}$ refers to a particular element of the matrix R_{rf}.

The notation makes sense if we consider that in three degrees of freedom, the elements of the phase space vector are the variables x, p_x, y, p_y, z and δ.

Thus, δ and z are the 6$^{\text{th}}$ and 5$^{\text{th}}$ elements (respectively) of the phase space vector.

By convention, coefficients of linear terms are denoted R_{ij}, coefficients of second-order terms are denoted T_{ijk}, third-order terms U_{ijkl} and so on:

$$\begin{aligned} x_i(s_1) = \quad & \sum_j R_{ij} x_j(s_0) \\ & + \sum_{j,k} T_{ijk} x_j(s_0) x_k(s_0) \\ & + \sum_{j,k,l} U_{ijkl} x_j(s_0) x_k(s_0) x_l(s_0) \ldots \end{aligned}$$

The co-ordinates of the particle after the chicane are then (again to second order):

$$z_3 = z_1 + (R_{\text{ch}})_{56}\delta_1 + (T_{\text{ch}})_{566}\delta_1^2, \tag{25}$$

$$\delta_3 = \delta_1. \tag{26}$$

In the present case, to simplify the notation we can drop the subscripts "rf" and "ch": since the maps for the rf cavity and the chicane involve different elements of the matrices R_{rf} and R_{ch} (and T_{rf} and T_{ch}), there is no ambiguity.

If we combine the maps for the rf and the chicane, we get:

$$\begin{aligned} z_3 &= (1 + R_{56}R_{65})z_0 + R_{56}\delta_0 \\ &\quad + (R_{56}T_{655} + R_{65}^2 T_{566})z_0^2 \\ &\quad + 2R_{65}T_{566}z_0\delta_0 \\ &\quad + T_{566}\delta_0^2, \end{aligned} \tag{27}$$

$$\delta_3 = \delta_0 + R_{65}z_0 + T_{655}z_0^2. \tag{28}$$

The term that gives the strong nonlinear distortion is the term in z_0^2 in (27). If we can design a system such that the appropriate coefficients satisfy:

$$R_{56}T_{655} + R_{65}^2 T_{566} = 0, \tag{29}$$

then we should be able to reduce the distortion.

The values of $R_{56} = b$ and $R_{65} = -a$ are determined by the requirements for the compression factor.

The value of T_{566} is determined by the chicane; in fact, we find for $\theta_0 \ll 1$ (see Exercise 3):

$$T_{566} \approx -3L_1\theta_0^2 \approx -\frac{3}{2}R_{56}. \tag{30}$$

Our only degree of freedom is with the coefficient T_{655}: this is the second-order dependence of the energy deviation on longitudinal position for a particle passing through the rf cavity.

Unfortunately, if we inspect the full rf map (3), we find it contains only odd-order terms, so $T_{655} = 0$. However...

...we can operate the rf cavity off-phase. Then, the rf map becomes:

$$z_1 = z_0, \tag{31}$$
$$\delta_1 = \delta_0 - \frac{eV}{E_0} \sin\left(\frac{\omega z_0}{c} + \phi_0\right). \tag{32}$$

The first-order coefficient in the map for δ is then:

$$R_{65} = -\frac{eV}{E_0}\frac{\omega}{c}\cos\phi_0. \tag{33}$$

The second-order coefficient is:

$$T_{655} = \frac{1}{2}\frac{eV}{E_0}\left(\frac{\omega}{c}\right)^2 \sin\phi_0. \tag{34}$$

Note that there is also a zeroth-order term, so the bunch ends up with a non-zero mean energy deviation $\langle\delta\rangle$ after the rf cavity; but we can take this into account simply by an appropriate scaling of the field in the chicane.

The map for the ILC bunch compressor now has the following coefficients.

The linear coefficients are determined by the required compression factor, and the requirement to have no final correlation $\langle z\delta\rangle$:

$$R_{65} = -4.9937\,\mathrm{m}^{-1}, \quad \text{and} \quad R_{56} = 0.19975\,\mathrm{m}. \tag{35}$$

The value of T_{566} is determined by the R_{56} of the chicane:

$$T_{566} = -\frac{3}{2}R_{56} = -0.29963\,\mathrm{m}. \tag{36}$$

And the value of T_{655} is determined by the desire to correct the second-order distortion of the phase space:

$$R_{56}T_{655} + R_{65}^2 T_{566} = 0 \quad \therefore \quad T_{655} = 37.406\,\mathrm{m}^{-2}. \tag{37}$$

Now, given:

$$R_{65} = -\frac{eV}{E_0}\frac{\omega}{c}\cos\phi_0 = -4.9937\,\mathrm{m}^{-1}, \tag{38}$$

and:

$$T_{655} = \frac{1}{2}\frac{eV}{E_0}\left(\frac{\omega}{c}\right)^2 \sin\phi_0 = 37.406\,\mathrm{m}^{-2}, \tag{39}$$

we find, for $E_0 = 5\,\mathrm{GeV}$ and $\omega = 1.3\,\mathrm{GHz}$:

$$V = 1,046\,\mathrm{MV}, \quad \text{and} \quad \phi_0 = 28.8^\circ. \tag{40}$$

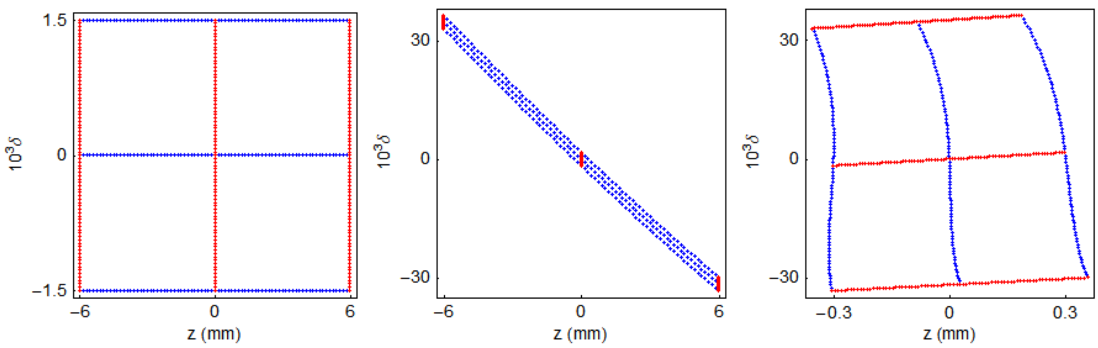
Operating with this phase, we are providing over a gigavolt of rf to *decelerate* the beam by more than 500 MV.

Because of adiabatic (anti)damping, we will need to reduce the R_{56} of the chicane by a factor E_1/E_0, where E_0 and E_1 are the mean bunch energy before and after the rf, respectively.

Also, the phase space area occupied by the distribution will be *increased* by a factor E_0/E_1.

As usual, we illustrate the effect of the bunch compressor on phase space using a "window frame" distribution.

Using the latest set of parameters, we find the following:



This looks much better: the dominant distortion now appears to be third-order, and looks small enough that it may not significantly affect the performance of the collider (although this would need to be checked by more detailed studies).

We have already learned some important lessons from this example:

- Ignoring nonlinear effects can get you into trouble. Sometimes you can get away with it; other times, a system designed without taking into account nonlinearities will not achieve the specified performance.

- If we take the trouble to analyse and understand the nonlinear behaviour of a system, then, if we are lucky enough and clever enough, we may be able to devise a means of compensating any adverse effects.

1. Derive the expressions for the parameters a and b given in equation (13).

2. Show that the transfer matrix R_{bc} given in (14) obeys the symplectic condition (16) for any values of the parameters a and b.

3. Show that for a chicane constructed from four dipoles, the Taylor map coefficients R_{56} and T_{566} are related by:

$$T_{566} = -\frac{3}{2}R_{56}.$$

4. Show that the transfer map for a bunch compressor (including RF and chicane) is given (to second order in the dynamical variables) by equations (27) and (28).