LONGITUDINAL DYNAMICS IN PARTICLE ACCELERATORS

by

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Cockroft Institute, November 2009

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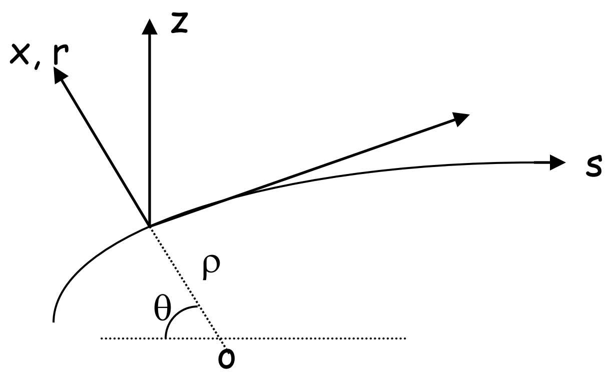
Acceleration & Curvature



Within the assumption:

$$\vec{E} \rightarrow E_\theta$$

$$\vec{B} \rightarrow B_z$$

the Newton-Lorentz force:

$$\frac{d\vec{p}}{dt} = e\vec{E} + e\vec{v} \times \vec{B}$$

becomes:

$$\frac{d(mv_\theta)}{dt}\vec{u}_\theta - m\frac{v_\theta^2}{\rho}\vec{u}_r = eE_\theta\vec{u}_\theta - ev_\theta B_z\vec{u}_r$$

leading to:

$$\frac{dp_\theta}{dt} = eE_\theta$$

$$\frac{p_\theta}{e} = B_z\rho$$

Energy Gain

In relativistic dynamics, energy and momentum satisfy the relation:

$$E^2 = E_0^2 + p^2c^2 \qquad \left(E = E_0 + W\right)$$

Hence:

$$dE = v\,dp$$

The rate of energy gain per unit length of acceleration (along z) is then:

$$\frac{dE}{dz} = v\frac{dp}{dz} = \frac{dp}{dt} = eE_z$$

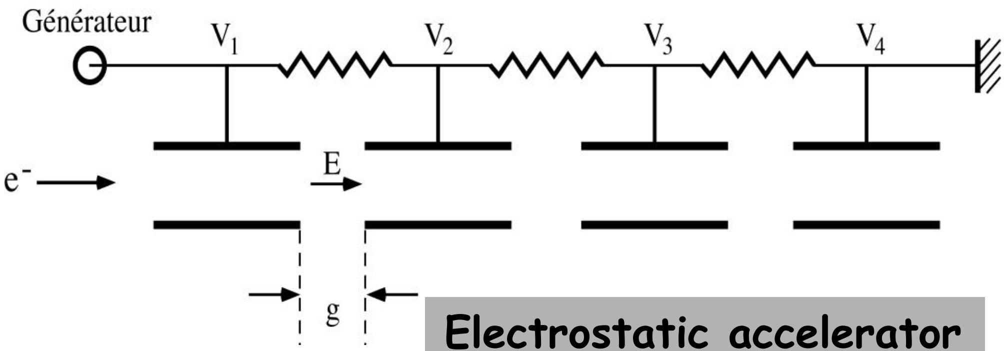
and the kinetic energy gained from the field along the z path is:

$$dW = dE = eE_z dz \qquad \Rightarrow \qquad W = e\int E_z dz = eV$$

where V is just a potential

Methods of Acceleration

1_ Electrostatic Field

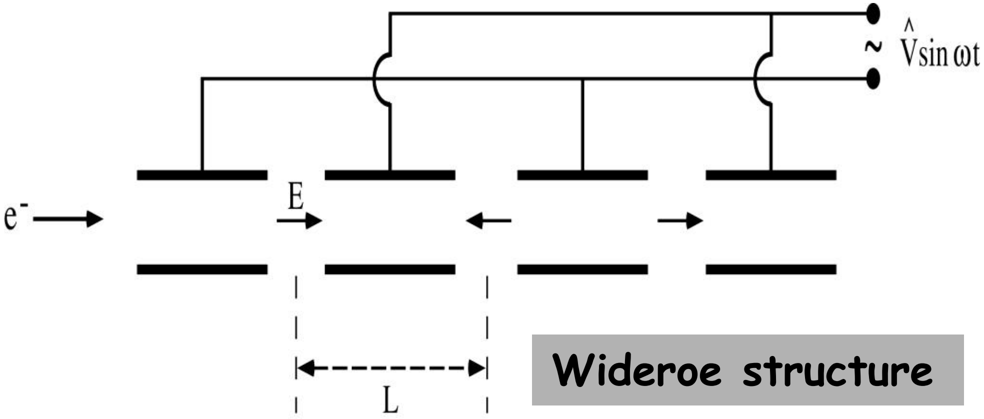


Electrostatic accelerator

Energy gain : $W=n.e(V_2-V_1)$

limitation : $V_{generator}=\Sigma V_i$

2_ Radio-frequency Field



Wideroe structure

Synchronism : ➡ $L=vT/2$

v=particle velocity **T= RF period**

also : $$L = v\frac{T}{2} = \beta\frac{\lambda_0}{2}$$

Methods of Acceleration (2)

3_ Acceleration by induction

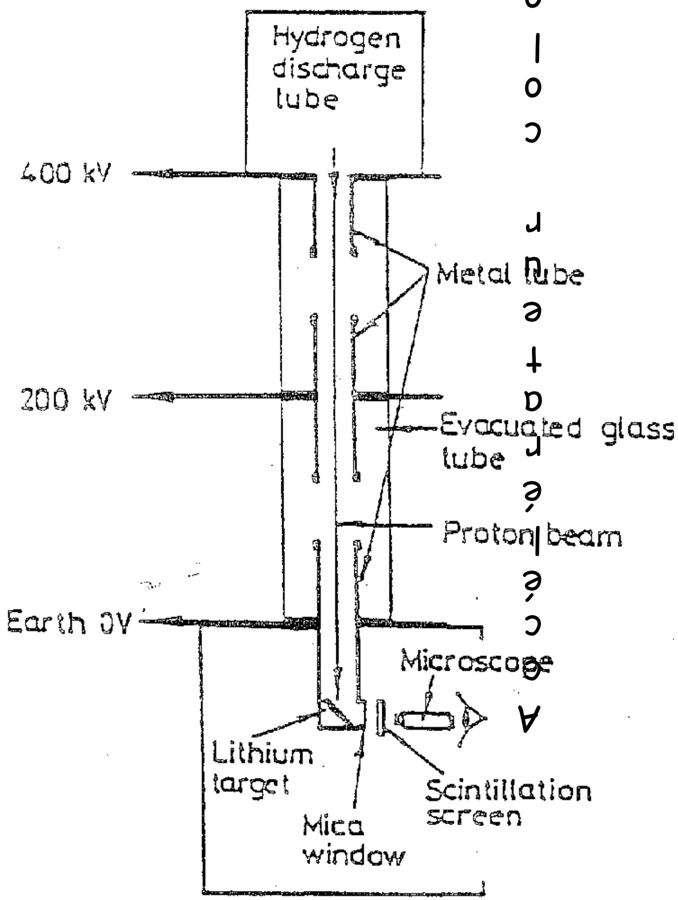
From MAXWELL EQUATIONS :

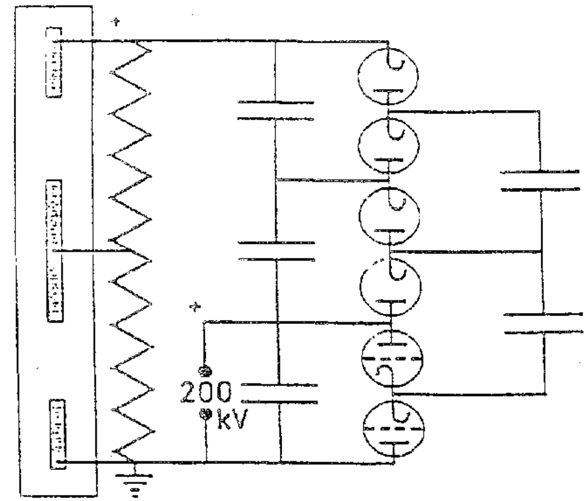
The electric field is derived from a scalar potential ϕ and a vector potential **A**
The time variation of the magnetic field **H** generates an electric field **E**

$$\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}$$

$$\vec{B} = \mu\vec{H} = \vec{\nabla} \times \vec{A}$$

Electrostatic accelerator





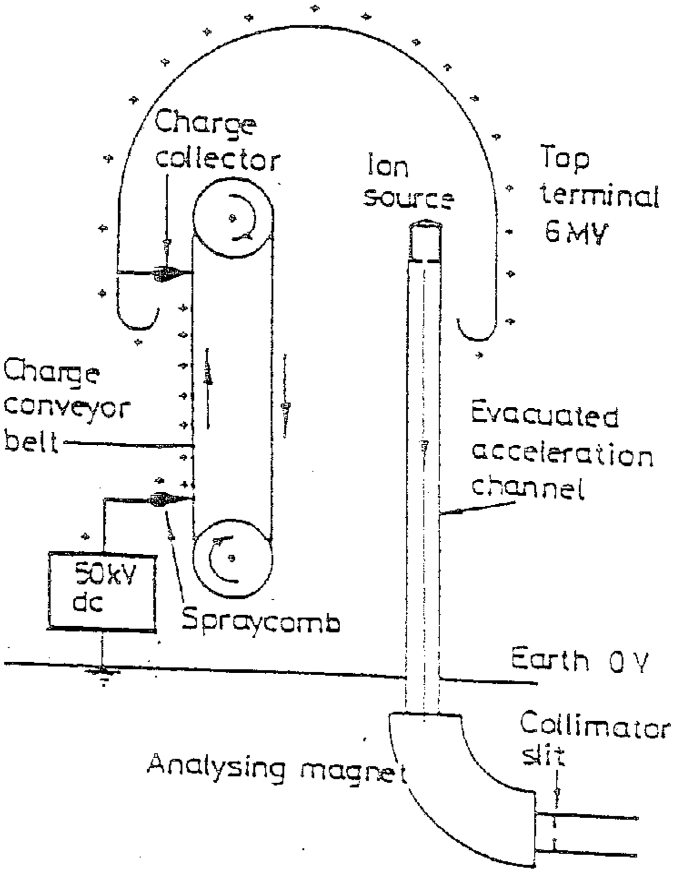
d.c. high voltage generator

Accelerating column

Accelerating column

HV system used by Cockcroft & Walton to break the lithium nucleus

Electrostatic accelerator (2)



An insulated belt is used to transport electric charges to a HV terminal .

The charges are generated by field effect from a comb on the belt . At the terminal they are extracted in a similar way.

The HV is distributed along the column through a resistor.

Van de Graaf type electrostatic accelerator

Betatron

Induction law

$$2\pi R E_\theta = -\frac{d\Phi}{dt} = -\pi R^2 \frac{d\overline{B_z}}{dt}$$

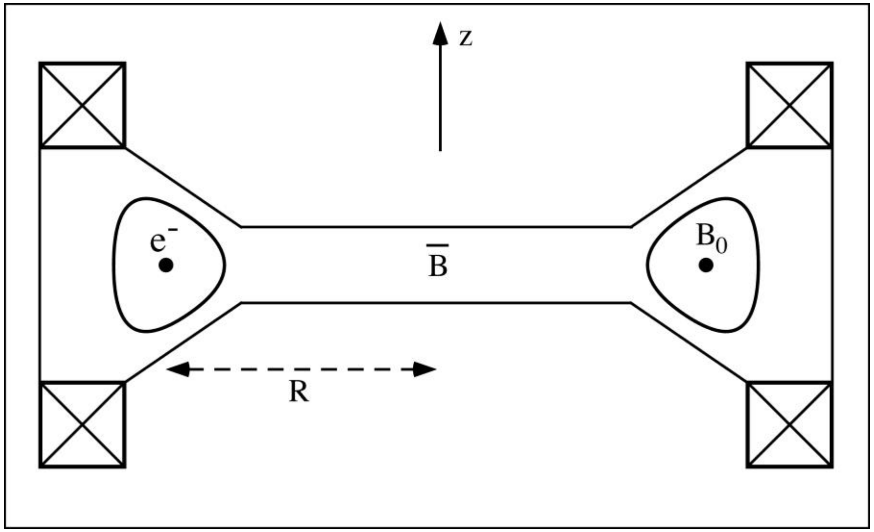
Newton-Lorentz force

$$\frac{dp}{dt} = eE_\vartheta = -\frac{1}{2} e R \frac{d\overline{B_z}}{dt}$$

A constant trajectory also requires :

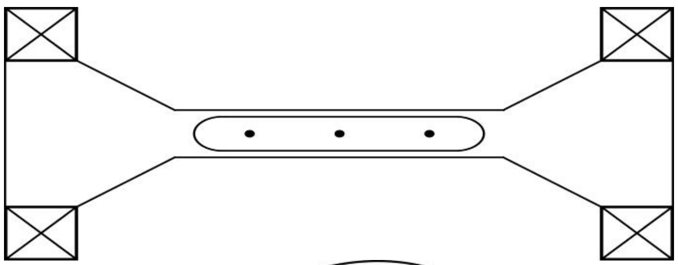
$$p = -e R B_0$$

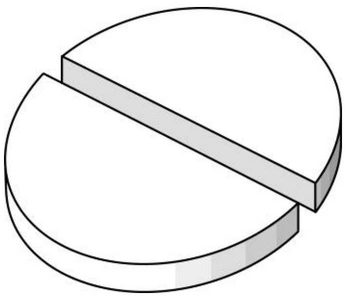
$$\frac{dp}{dt} = -e R \frac{dB_0}{dt} \Rightarrow \boxed{B_o = \frac{1}{2}\overline{B_z}}$$

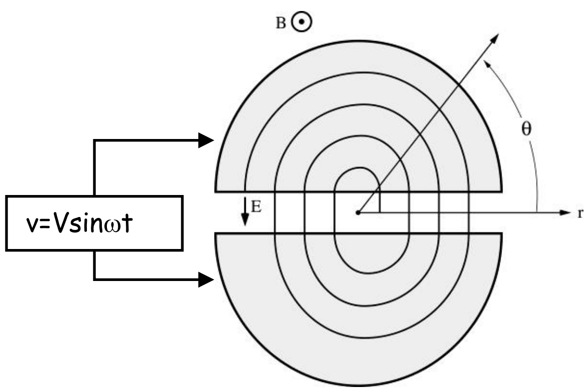


The betatron uses a variable magnetic field with time. The pole shaping gives a magnetic field Bo at the location of the trajectory, smaller than the average magnetic field.

Cyclotron







At each radius r corresponds a velocity v for the accelerated particle. The half circle corresponds to half a revolution period T/2 and B is constant:

$$r = \frac{p}{eB} = \frac{mv}{eB} \Longrightarrow \frac{T}{2} = \frac{\pi m}{eB}$$

The corresponding angular frequency is :

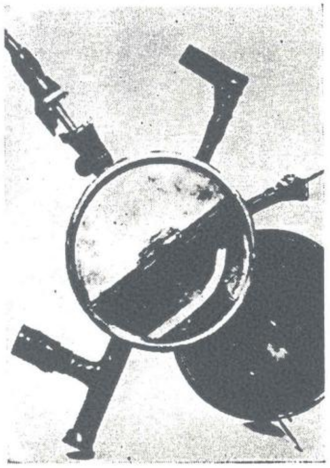
$$\omega_r = 2\pi f_r = \frac{2\pi}{T} = \frac{eB}{m}$$

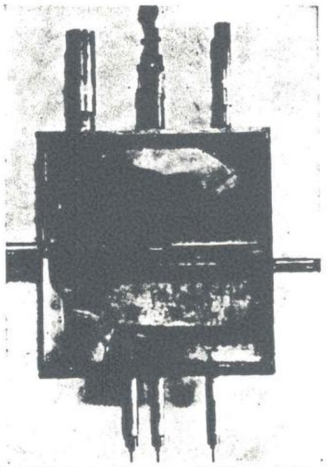
Synchronism if : $\boxed{\omega_{RF} = \omega_r}$

m = m_0 (constant) if W << E_0

If so the cyclotron is <u>isochronous</u>

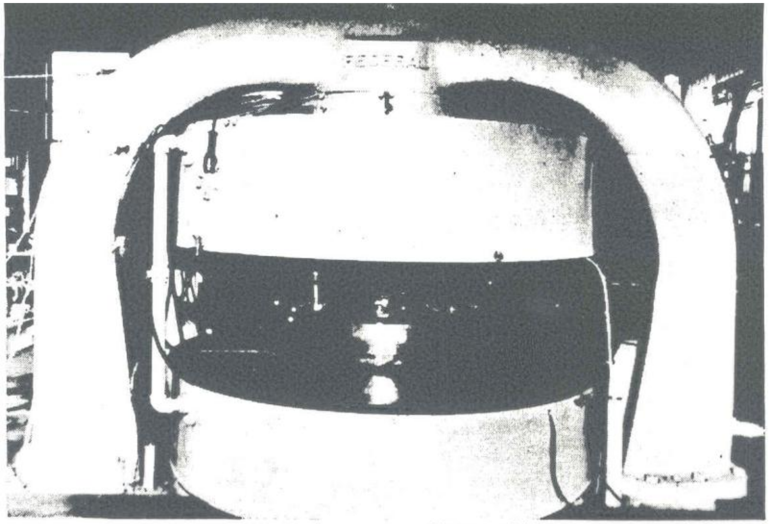
Cyclotron (2)





Both Courtesy of Lawrence Radiation Laboratory

Here below the 27-inch cyclotron, Berkeley (1932). The magnet was originally part of the resonant circuit of an RF current generator used in telecommunications.



Courtesy of Lawrence Radiation Laboratory

Cyclotron of M.S.Livingstone (1931)
On the left the 4-inch vacuum chamber
Used to validate the concept.
On the right the 11-inch vacuum chamber of the Berkeley cyclotron that produced 1,2 MeV protons.
In both cases one single electrode (dee).

Cyclotron (4)

Cyclotrons at GANIL, Caen

Cyclotron (5)

Energy-phase equation:

Energy gain at each gap transit: $$\Delta E = e\hat{V}\sin\phi$$

Particle RF phase versus time: $$\phi = \omega_{RF}t - \theta$$

where θ is the azimuthal angle of trajectory

Differentiating with respect to time gives: $$\dot{\phi} = \omega_{RF} - \omega_r = \omega_{RF} - ec^2\frac{B}{E}$$

Smooth approximation allows: $$\dot{\phi} = \frac{\Delta\phi}{T_r/2} = \frac{\omega_r}{\pi}\Delta\phi$$

Relative phase change at ½ revolution $$\Delta\phi = \frac{\pi}{\omega_r}\dot{\phi} = \pi\left(\frac{\omega_{RF}E}{ec^2B} - 1\right)$$

And smooth approximation again: $$\frac{d\phi}{dE} = \frac{\Delta\phi}{\Delta E} = \frac{\pi}{e\hat{V}\sin\phi}\left(\frac{\omega_{RF}E}{ec^2B} - 1\right)$$

Cyclotron (6)

Separating:

$$d(\cos\phi) = -\frac{\pi}{e\hat{V}}\left(\frac{\omega_{RF}E}{ec^2B} - 1\right)dE$$

Integrating:

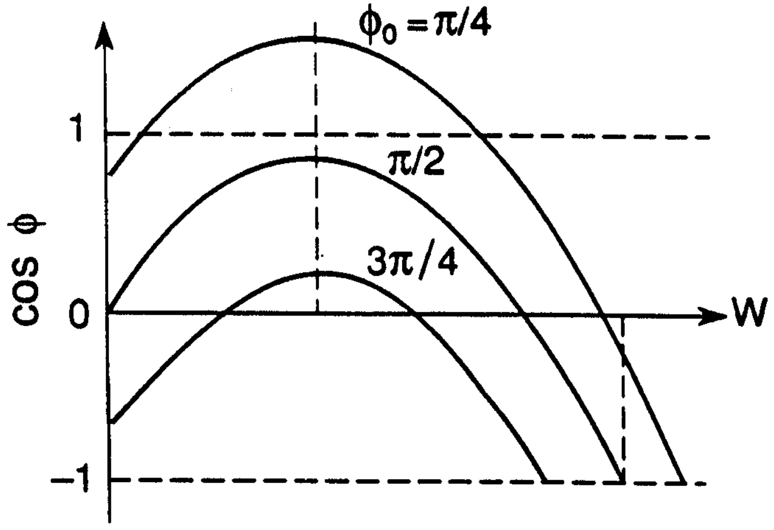
$$\cos\phi = \cos\phi_0 + \frac{\pi}{e\hat{V}}\left(1 - \frac{\omega_{RF}}{\omega_{r0}}\right)(E - E_0) - \frac{\pi}{2e\hat{V}E_0}\frac{\omega_{RF}}{\omega_{r0}}(E - E_0)^2$$

with :

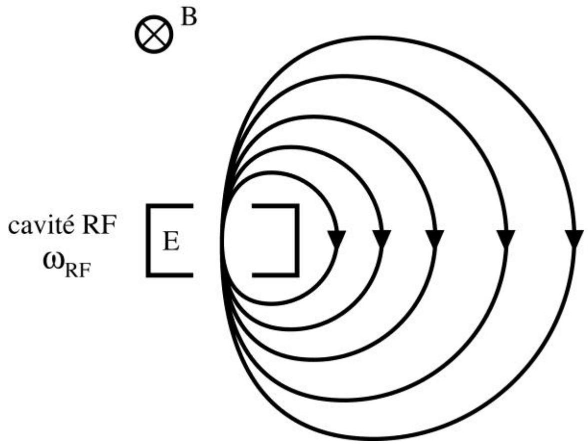
$E_0 =$ Rest energy

$\phi_0 =$ Injection phase

$\omega_{r0} =$ Starting revolution frequency



Microtron (Veksler, 1954)



The expression $\omega_r = \dfrac{eB}{m}$ shows that if the mass increases, the frequency decreases :

$m \nearrow \quad \Longrightarrow \quad \omega_r \searrow$

Synchronism condition: $T_r \propto m \propto \gamma$

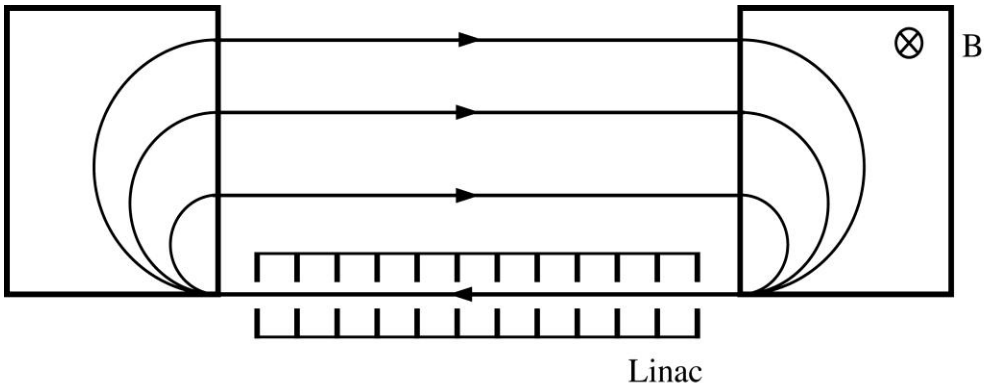
If the first turn is synchronous : $\dfrac{\Delta T_r}{T_{RF}} = \text{integer} \Rightarrow \Delta\gamma_{turn} = \text{integer}\,(\gamma_0 = 1)$

Energy gain per turn
- electrons $\rightarrow$ 0.511 MeV
- protons $\rightarrow$ 0.938 GeV !!!

Since required energy gains are large the concept is essentially valid for electrons.

Microtron « Racetrack »

Allows to increase the energy gain per turn by using several accelerating cavities (ex : linac section)



Synchronism is obtained when the energy gain per turn is a multiple of the rest energy:

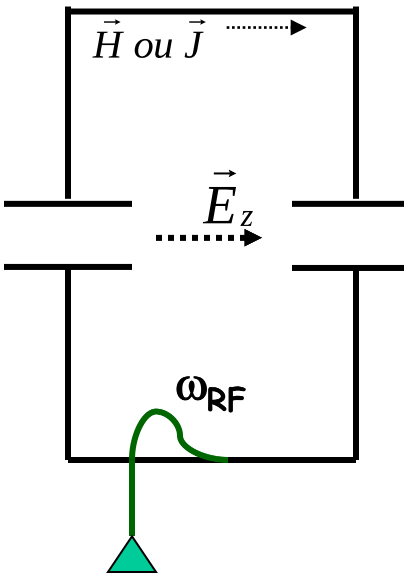
$$(\Delta\gamma)/turn = integer$$

Carefull !!!! This is not a « recirculating » linac

The advantage of Resonant Cavities

- Considering RF acceleration, it is obvious that when particles get high velocities the drift spaces get longer and one loses on the efficiency. The solution consists of using a higher operating frequency.

- The power lost by radiation, due to circulating currents on the electrodes, is proportional to the RF frequency. The solution consists of enclosing the system in a cavity which resonant frequency matches the RF generator frequency.

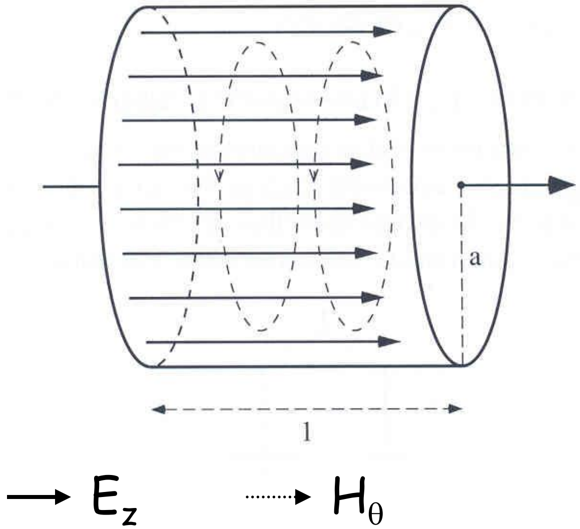


-Each such cavity can be independently powered from the RF generator.

- The electromagnetic power is now constrained in the resonant volume.

- Note however that joule losses will occur in the cavity walls (unless made of superconducting materials)

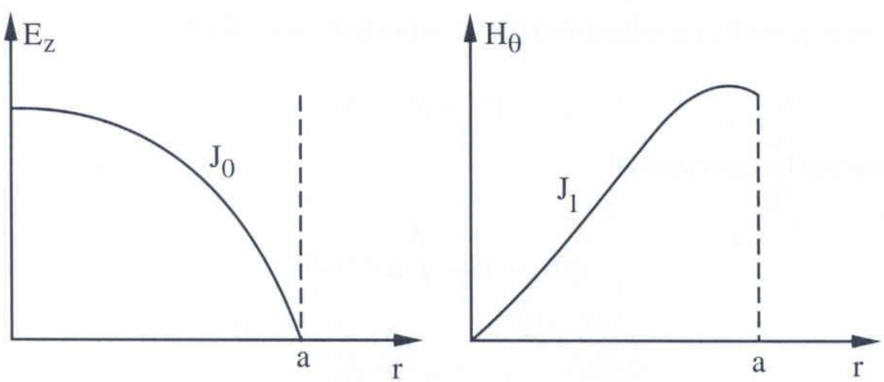
The Pill Box Cavity



From Maxwell's equations one can derive the wave equations :

$$\nabla^2 A - \varepsilon_0 \mu_0 \frac{\partial^2 A}{\partial t^2} = 0 \qquad (A = E \text{ ou } H)$$

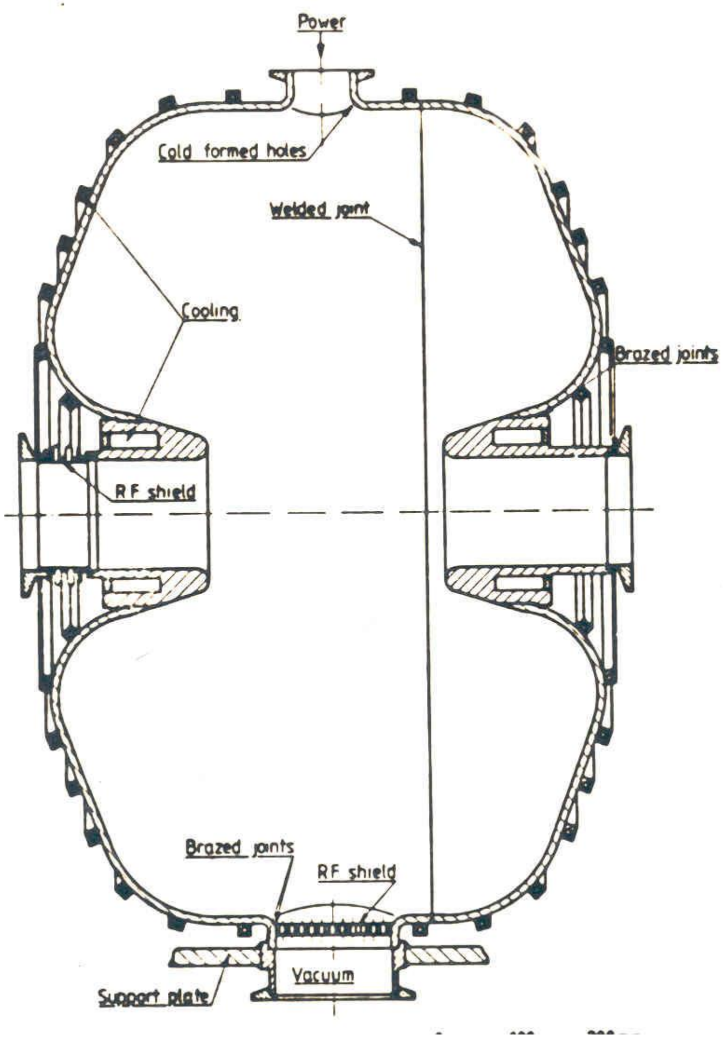
Solutions for E and H are oscillating modes, at discrete frequencies, of types TM ou TE. For l<2a the most simple mode, TM_{010}, has the lowest frequency ,and has only two field components:



$$\left.\begin{aligned} E_z &= J_0(kr) \\ H_\theta &= -\frac{j}{Z_0} J_1(kr) \end{aligned}\right\} e^{j\omega t}$$

$$k = \frac{2\pi}{\lambda} = \frac{\omega}{c} \quad \lambda = 2{,}62\,a \quad Z_0 = 377\Omega$$

The Pill Box Cavity (2)



The design of a pill-box cavity can be sophisticated in order to improve its performances:

-A nose cone can be introduced in order to concentrate the electric field around the axis,

-Round shaping of the corners allows a better distribution of the magnetic field on the surface and a reduction of the Joule losses. It also prevent from multipactoring effects.

A good cavity is a cavity which efficiently transforms the RF power into accelerating voltage.

Energy Gain with RF field

RF acceleration

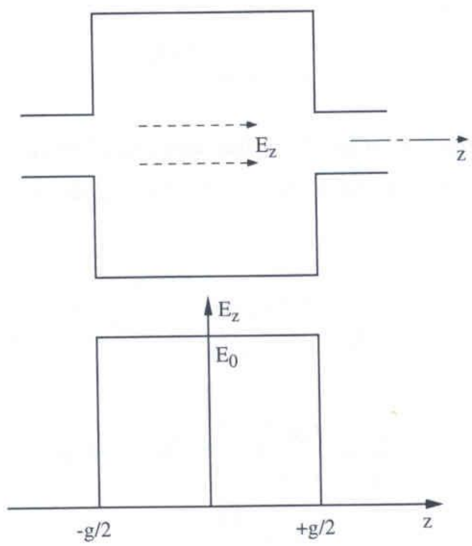
In this case the electric field is oscillating. So it is for the potential. The energy gain will depend on the RF phase experienced by the particle.

$$\int \hat{E}_z \, dz = \hat{V} \qquad E_z = \hat{E}_z \cos \omega_{RF} t \quad = \quad \hat{E}_z \cos \Phi(t)$$

$$W = e \hat{V} \cos \Phi$$

Neglecting the transit time in the gap.

Transit Time Factor



Oscillating field at frequency ω and which amplitude is assumed to be constant all along the gap:

$$E_z = E_0 \cos\omega t = \frac{V}{g}\cos\omega t$$

Consider a particle passing through the middle of the gap at time t=0 :

$$z = vt$$

The total energy gain is:

$$\Delta W = \frac{eV}{g}\int_{-g/2}^{g/2}\cos\omega\frac{z}{v}dz$$

$$\theta = \frac{\omega g}{v} \quad \textit{transit angle}$$

$$T \quad \textit{transit time factor}$$

(0 < T < 1)

$$\boxed{\Delta W = eV\frac{\sin\theta/2}{\theta/2} = eVT}$$

Transit Time Factor (2)

Consider the most general case and make use of complex notations:

$$\Delta E = e\Re_e \int_0^g E_z(z) e^{j\omega t} dz \qquad \omega t = \omega \frac{z}{v} - \psi_p$$

Ψ_p is the phase of the particle entering the gap with respect to the RF.

$$\Delta E = e\Re_e \left[e^{-j\psi_p} \int_0^g E_z(z) e^{j\omega \frac{z}{v}} dz \right]$$

$$\Delta E = e\Re_e \left[e^{-j\psi_p} e^{j\psi_i} \left| \int_0^g E_z(z) e^{j\omega \frac{z}{v}} dz \right| \right]$$

Introducing: $\phi = \psi_p - \psi_i$

$$\Delta E = e \left| \int_0^g E_z(z) e^{j\omega \frac{z}{v}} dz \right| \cos\phi$$

and considering the phase which yields the maximum energy gain:

$$T = \frac{\left| \int_0^g E_z(z) e^{j\omega t} dz \right|}{\int_0^g E_z(z) dz}$$

Important Parameters of Accelerating Cavities

Shunt Impedance

Relationship between gap voltage and wall losses.

$$P_d = \frac{V^2}{R}$$

Quality Factor

$$Q = \frac{\omega W_s}{P_d}$$

Relationship between stored energy in the volume and dissipated power on the walls.

$$\frac{R}{Q} = \frac{V^2}{\omega W_s}$$

Filling Time

$$P_d = -\frac{dW_s}{dt} = \frac{\omega}{Q} W_s$$

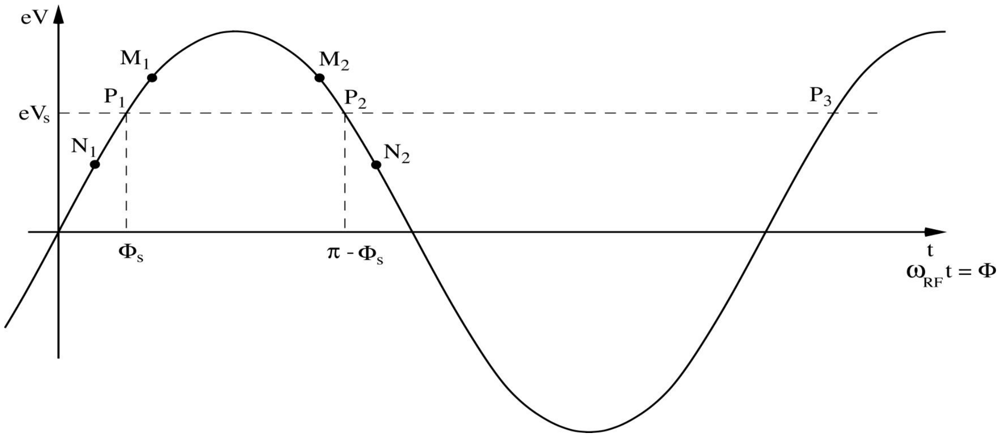
Exponential decay of the stored energy due to losses.

$$\tau = \frac{Q}{\omega}$$

Principle of Phase Stability

Let's consider a succession of accelerating gaps, operating in the 2π mode, for which the synchronism condition is fulfilled for a phase Φ_s .

For a 2π mode, the electric field is the same in all gaps at any given time.



$$eV_S = e\hat{V}\sin\Phi_S$$

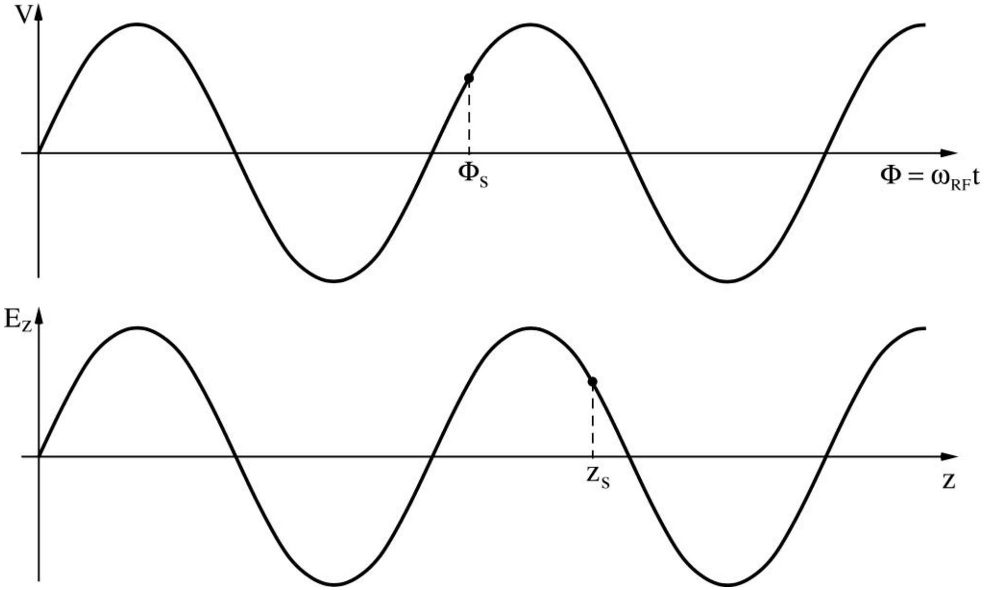
is the energy gain in one gap for the particle to reach the next gap with the same RF phase: P_1 ,P_2, …… are fixed points.

If an increase in energy is transferred into an increase in velocity, M_1 & N_1 will move towards P_1(stable), while M_2 & N_2 will go away from P_2 (unstable).

A Consequence of Phase Stability

Transverse Instability

Longitudinal phase stability means : $$\frac{\partial V}{\partial t} > 0 \Rightarrow \frac{\partial E_z}{\partial z} < 0$$

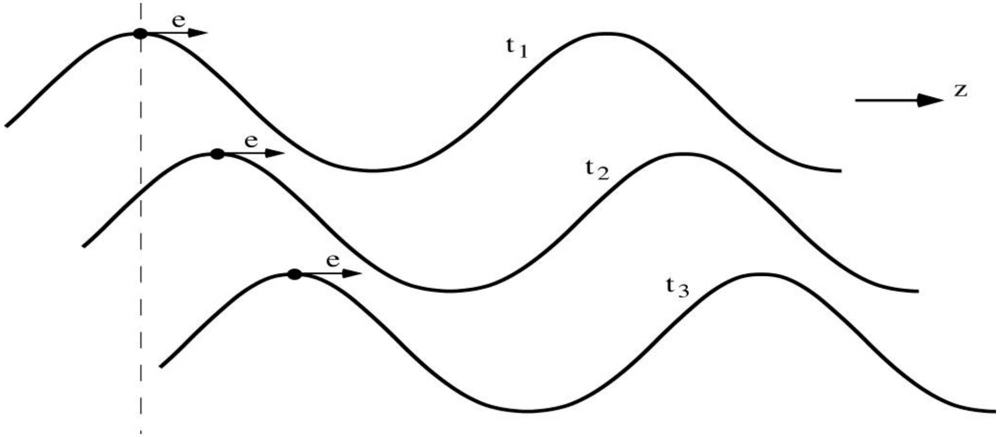


defocusing RF force

The divergence of the field is zero according to Maxwell : $$\nabla . \vec{E} = 0 \quad \Rightarrow \quad \frac{\partial E_x}{\partial x} + \frac{\partial E_z}{\partial z} = 0 \quad \Rightarrow \quad \frac{\partial E_x}{\partial x} > 0$$

External focusing (solenoid, quadrupole) is then necessary

The Traveling Wave Case



$$E_z = E_0 \cos(\omega_{RF} t - kz)$$

$$k = \frac{\omega_{RF}}{v_\varphi}$$

$$z = v(t - t_0)$$

$$v_\varphi = phase\ velocity$$

$$v = particle\ velocity$$

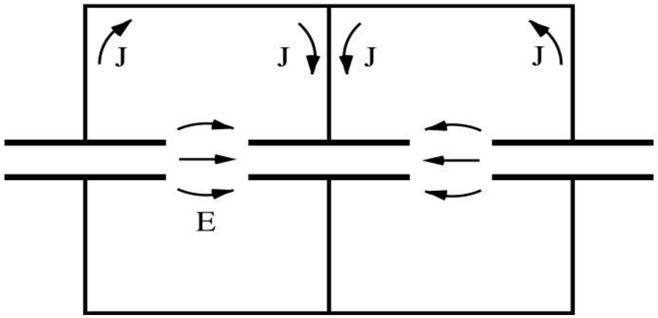
The particle travels along with the wave, and k represents the wave propagation factor.

$$E_z = E_0 \cos\left(\omega_{RF} t - \omega_{RF} \frac{v}{v_\varphi} t - \phi_0\right)$$

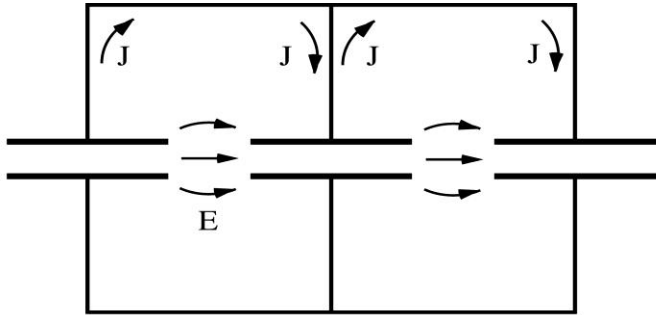
If synchronism satisfied: $v = v_\varphi \quad and \quad E_z = E_0 \cos \phi_0$

where ϕ_0 is the RF phase seen by the particle.

Multi-gaps Accelerating Structures:
A- Low Kinetic Energy Linac (protons,ions)

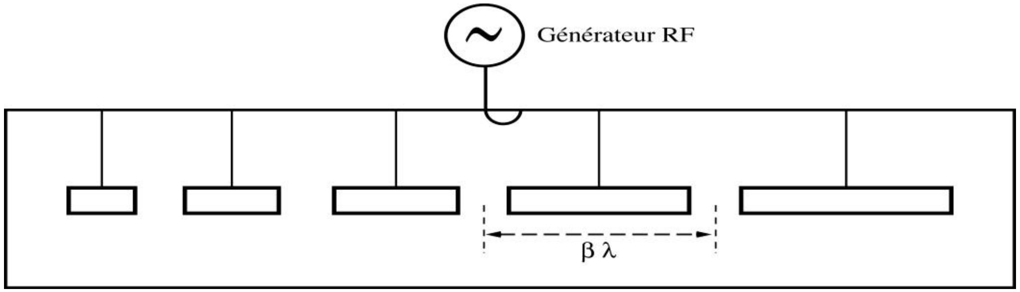


Mode π $L = vT/2$



Mode 2π $L = vT = \beta\lambda$

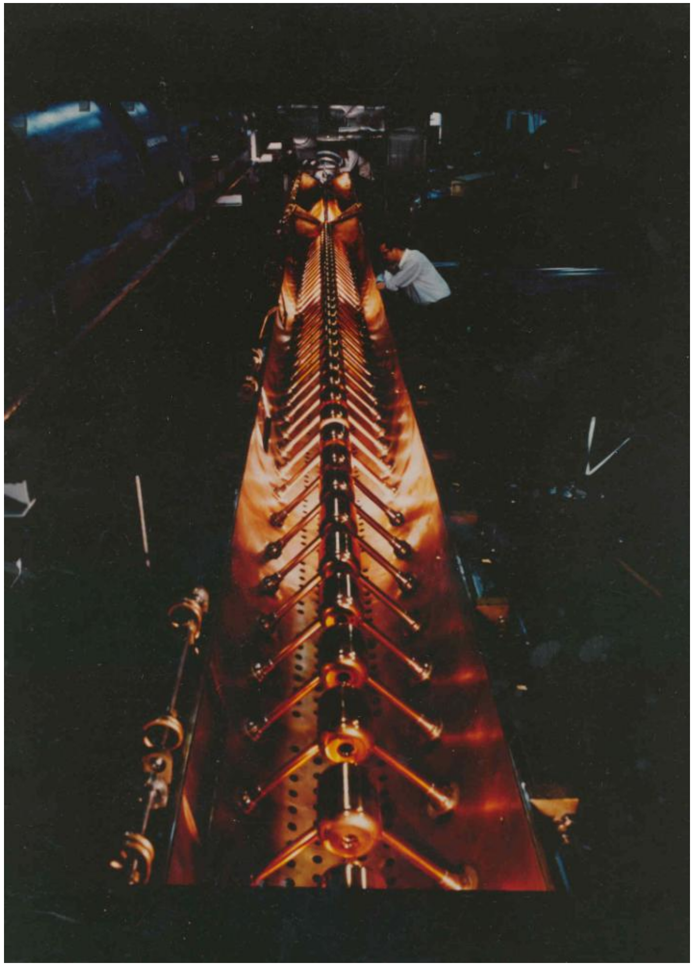
In « WIDEROE » structure radiated power $\propto \omega$ CV



ALVAREZ structure

In order to reduce the radiated power the gap is enclosed in a resonant volume at the operating frequency. A common wall can be suppressed if no circulating current in it for the chosen mode.

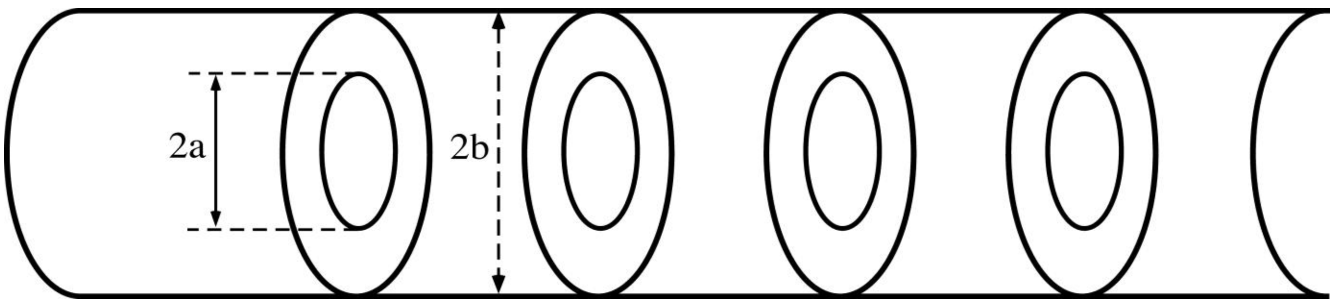
CERN Proton Linac



Multi-gaps Accelerating Structures:
B- High Energy Electron Linac

- When particles gets ultra-relativistic (v~c) the drift tubes become very long unless the operating frequency is increased. Late 40's the development of radar led to high power transmitters (klystrons) at very high frequencies (3 GHz).

- Next came the idea of suppressing the drift tubes using traveling waves. However to get a continuous acceleration the phase velocity of the wave needs to be adjusted to the particle velocity.



solution: slow wave guide with irises ⟶ iris loaded structure

Iris Loaded Structure for Electron Linac

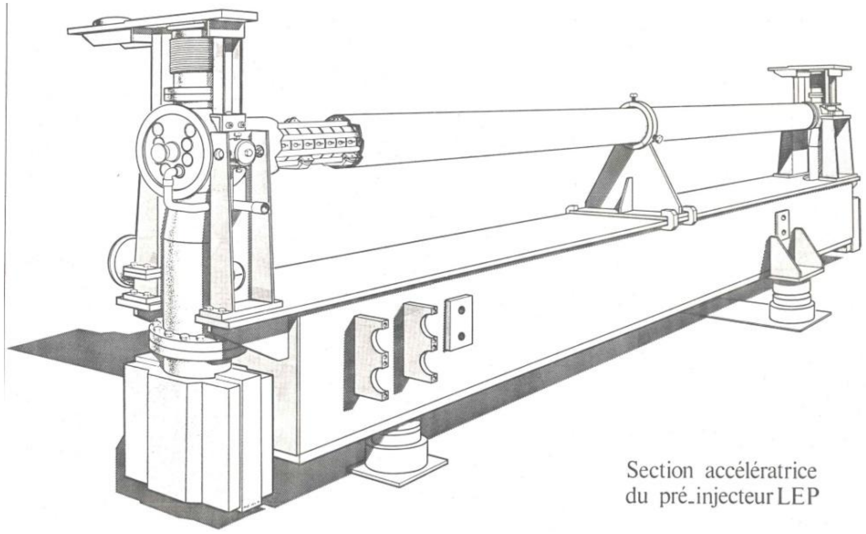
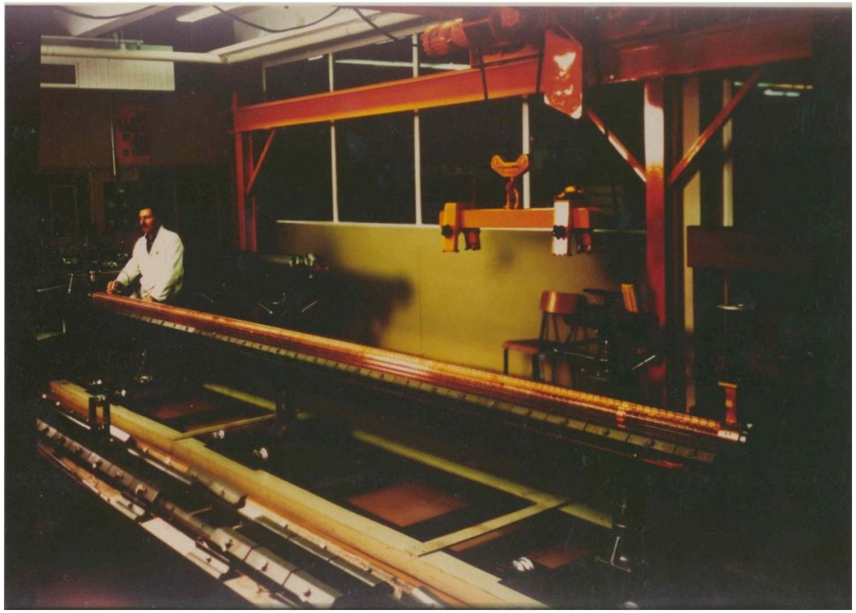


Photo of a CGR-MeV structure



4.5 m long copper structure, equipped with matched input and output couplers. Cells are low temperature brazed and a stainless steel envelope ensures proper vacuum.

Energy-phase Equations

- Rate of energy gain for the synchronous particle:

$$\frac{dE_s}{dz} = \frac{dp_s}{dt} = eE_0 \sin\phi_s$$

- Rate of energy gain for a non-synchronous particle, expressed in reduced variables, $w = W - W_s = E - E_s$ and $\varphi = \phi - \phi_s$:

$$\frac{dw}{dz} = eE_0\left[\sin(\phi_s + \varphi) - \sin\phi_s\right] \approx eE_0 \cos\phi_s . \varphi \quad (small\ \varphi)$$

- Rate of change of the phase with respect to the synchronous one:

$$\frac{d\varphi}{dz} = \omega_{RF}\left(\frac{dt}{dz} - \left(\frac{dt}{dz}\right)_s\right) = \omega_{RF}\left(\frac{1}{v} - \frac{1}{v_s}\right) \cong -\frac{\omega_{RF}}{v_s^2}(v - v_s)$$

Since:

$$v - v_s = c(\beta - \beta_s) \cong \frac{c}{2\beta_s}\left(\beta^2 - \beta_s^2\right) \cong \frac{w}{m_0 v_s \gamma_s^3}$$

Energy-phase Oscillations

one gets:

$$\frac{d\varphi}{dz} = -\frac{\omega_{RF}}{m_0 v_s^3 \gamma_s^3} w$$

Combining the two first order equations into a second order one:

$$\frac{d^2\varphi}{dz^2} + \Omega_s^2 \varphi = 0$$

with

$$\Omega_s^2 = \frac{eE_0 \omega_{RF} \cos \phi_s}{m_0 v_s^3 \gamma_s^3}$$

Stable harmonic oscillations imply:

$$\Omega_s^2 > 0 \quad \textit{and real}$$

hence: $\cos \phi_s > 0$

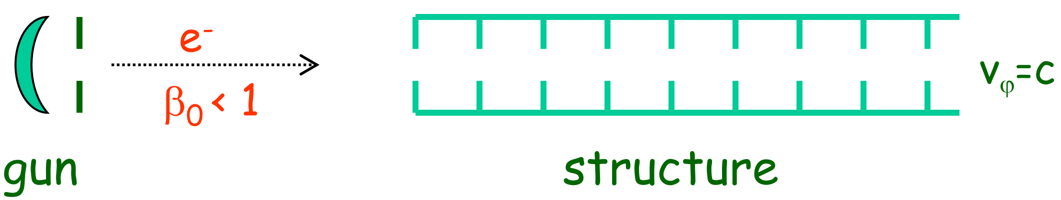
And since acceleration also means: $\sin \phi_s > 0$

One finally gets the results: $0 < \phi_s < \dfrac{\pi}{2}$

The Capture Problem

- Previous results show that at ultra-relativistic energies (γ >> 1) the longitudinal motion is frozen. Since this is rapidly the case for electrons, all traveling wave structures can be made identical (phase velocity=c).

- Hence the question is: can we capture low kinetic electrons energies (γ < 1), as they come out from a gun, using an iris loaded structure matched to c ?



$$E_z = E_0 \sin\phi(t)$$

The electron entering the structure, with velocity v < c, is not synchronous with the wave. The path difference, after a time dt, between the wave and the particle is:

$$dz = (c - v)dt$$

Since: $\phi = \omega_{RF} t - kz \quad with\ \ propagation\ \ factor\ \ k = \dfrac{\omega_{RF}}{v_\varphi} = \dfrac{\omega_{RF}}{c}$

one gets: $dz = \dfrac{c}{\omega_{RF}} d\phi = \dfrac{\lambda_g}{2\pi} d\phi$ and $\dfrac{d\phi}{dt} = \dfrac{2\pi}{\lambda_g} c(1 - \beta)$

The Capture Problem (2)

From Newton-Lorentz:

$$\frac{d}{dt}(mv) = m_0 c \frac{d}{dt}(\beta\gamma) = m_0 c \frac{d}{dt}\left(\frac{\beta}{\left(1-\beta^2\right)^{\frac{1}{2}}}\right) = eE_0 \sin\phi$$

Introducing a suitable variable: $\beta = \cos\alpha$

the equation becomes: $$\frac{d\alpha}{dt} = -\frac{eE_0}{m_0 c}\sin\phi \sin^2\alpha$$

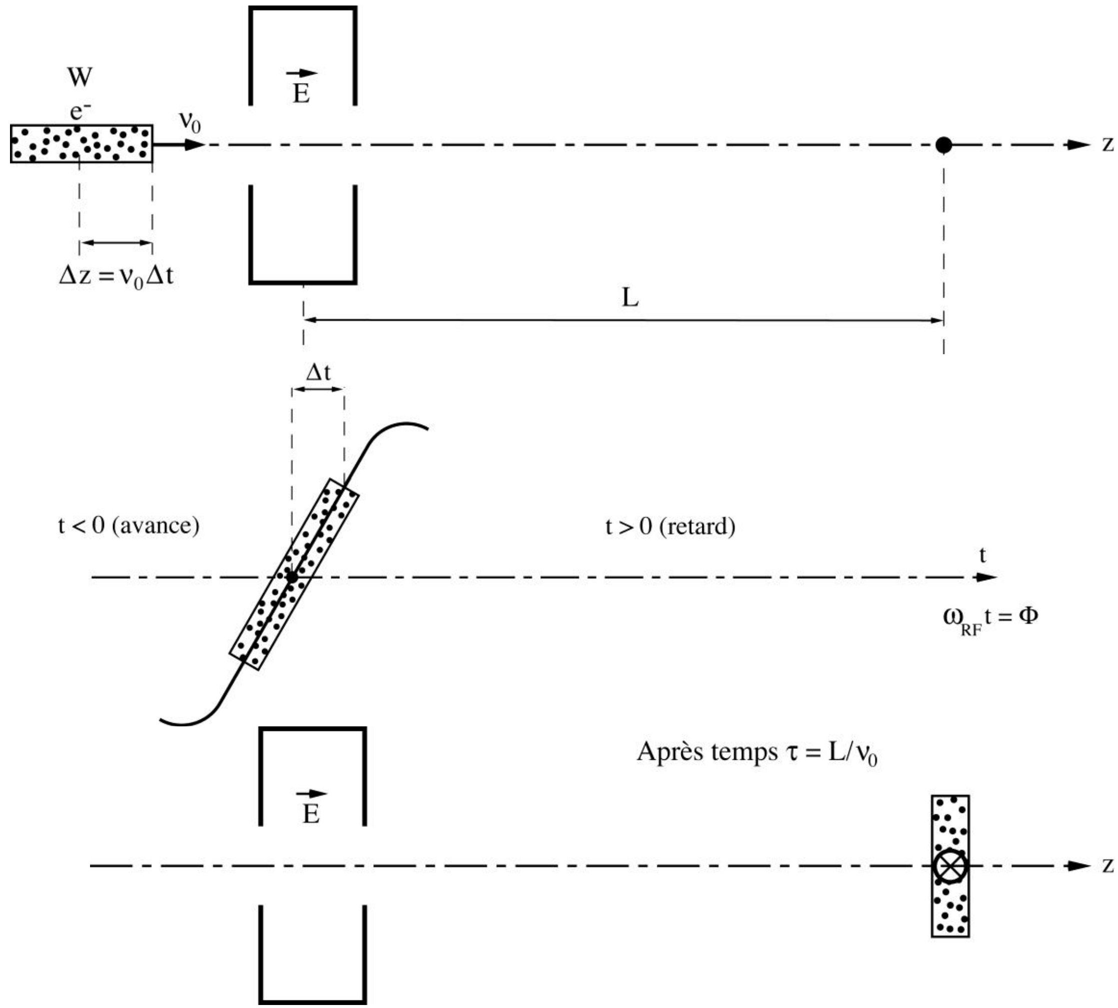
Using $\frac{d\phi}{dt} = \frac{d\phi}{d\alpha}\frac{d\alpha}{dt}$ $\longrightarrow$ $$-\sin\phi\, d\phi = \frac{2\pi m_0 c^2}{\lambda_g e E_0}\frac{1-\cos\alpha}{\sin^2\alpha}d\alpha$$

Integrating from t_0 to t (from $\beta=\beta_0$ to $\beta=1$) $\longrightarrow$ $$\cos\phi_0 - \cos\phi = \frac{2\pi m_0 c^2}{e\lambda_g E_0}\left(\frac{1-\beta_0}{1+\beta_0}\right)^{\frac{1}{2}} \le 2$$

Capture condition $\longrightarrow$ $$E_0 \ge \frac{\pi m_0 c^2}{e\lambda_g}\left(\frac{1-\beta_0}{1+\beta_0}\right)^{\frac{1}{2}}$$

Improved Capture With Pre-buncher

A long bunch coming from the gun enters an RF cavity; the reference particle is the one which has no velocity change. The others get accelerated or decelerated. After a distance L bunch gets shorter while energies are spread: bunching effect. This short bunch can now be captured more efficiently by a TW structure (v_φ=c).



Improved Capture With Pre-buncher (2)

The bunching effect is a space modulation that results from a velocity modulation and is similar to the phase stability phenomenon. Let's look at particles in the vicinity of the reference one and use a classical approach.

Energy gain as a function of cavity crossing time:

$$\Delta W = \Delta\left(\frac{1}{2} m_0 v^2\right) = m_0 v_0 \Delta v = eV_0 \sin\phi \approx eV_0 \phi$$

$$\Delta v = \frac{eV_0 \phi}{m_0 v_0}$$

Perfect linear bunching will occur after a time delay τ, corresponding to a distance L, when the path difference is compensated between a particle and the reference one:

$$\Delta v . \tau = \Delta z = v_0 t = v_0 \frac{\phi}{\omega_{RF}}$$

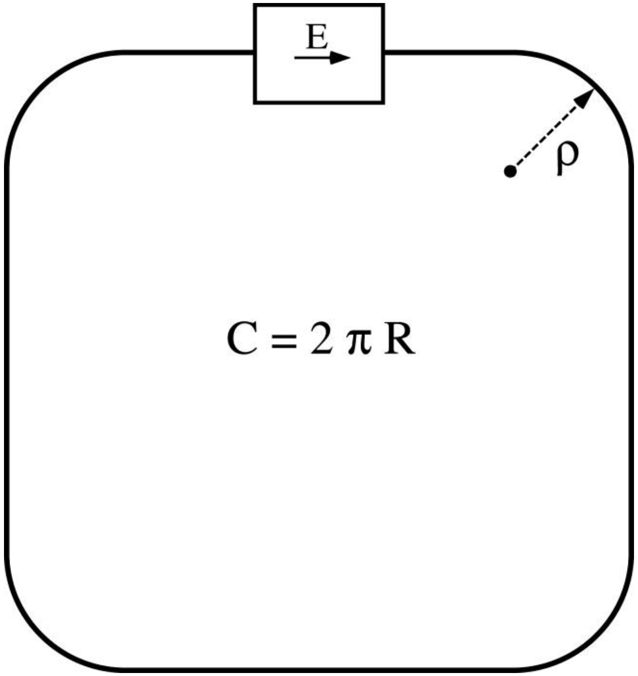
(assuming the reference particle enters the cavity at time t=0)

Since L = vτ one gets:

$$L = \frac{2 v_0 W}{eV_0 \omega_{RF}}$$

The Synchrotron (Mac Millan, Veksler, 1945)

The synchrotron is a synchronous accelerator since there is a synchronous RF phase for which the energy gain fits the increase of the magnetic field at each turn. That implies the following operating conditions:



$e\hat{V}\sin\Phi \longrightarrow$ Energy gain per turn

$\Phi = \Phi_s = cte \longrightarrow$ Synchronous particle

$\omega_{RF} = h\omega_r \longrightarrow$ RF synchronism

$\rho = cte \quad R = cte \longrightarrow$ Constant orbit

$B\rho = P/e \Rightarrow B \longrightarrow$ Variable magnetic field

If v = c, ω_r hence ω_{RF} remain constant (ultra-relativistic e^-)

The Synchrotron (2)

Energy ramping is simply obtained by varying the B field:

$$p = eB\rho \quad \Rightarrow \quad \frac{dp}{dt} = e\rho\, B' \quad \Rightarrow (\Delta p)_{turn} = e\rho\, B' T_r = \frac{2\pi\, e\rho\, RB'}{v}$$

Since: $$E^2 = E_0^2 + p^2 c^2 \quad \Rightarrow \quad \Delta E = v\Delta p$$

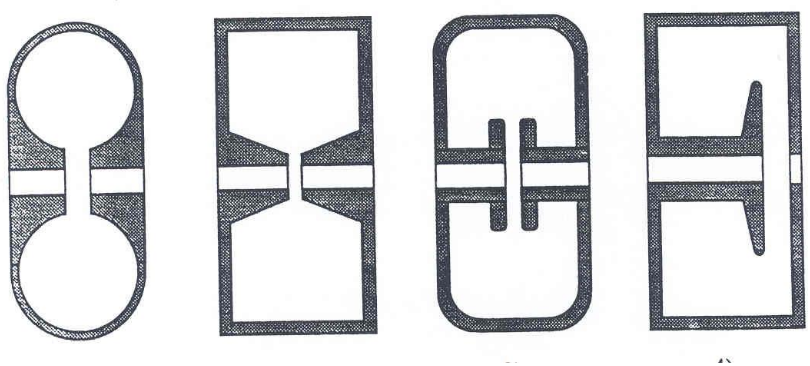
$$\left(\Delta E\right)_{turn} = \left(\Delta W\right)_s = 2\pi e\rho RB' = e\hat{V}\sin\phi_s$$

- The number of stable synchronous particles is equal to the harmonic number h. They are equally spaced along the circumference.
- Each synchronous particle satifies the relation p=eBρ. They have the nominal energy and follow the nominal trajectory.

Single Gap Types Cavities

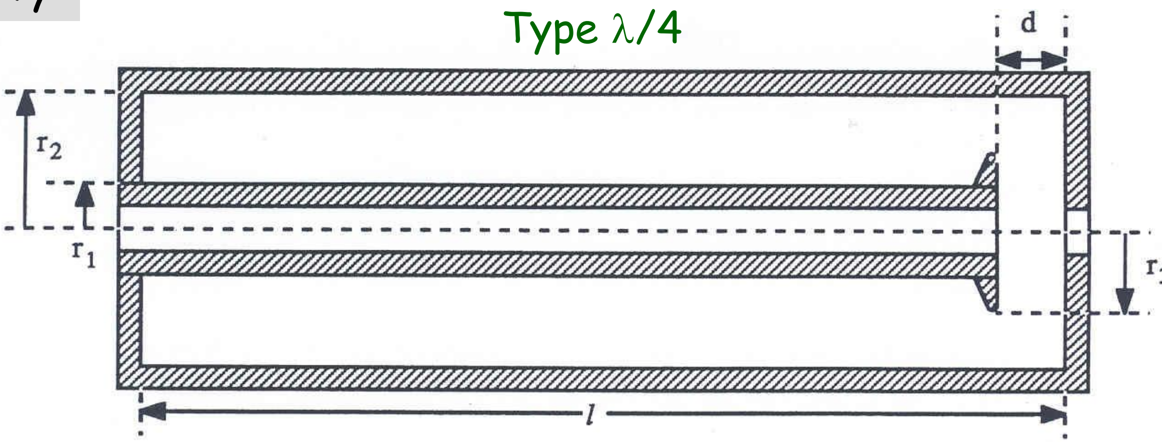
Pill-box variants

noses disks



Coaxial cavity

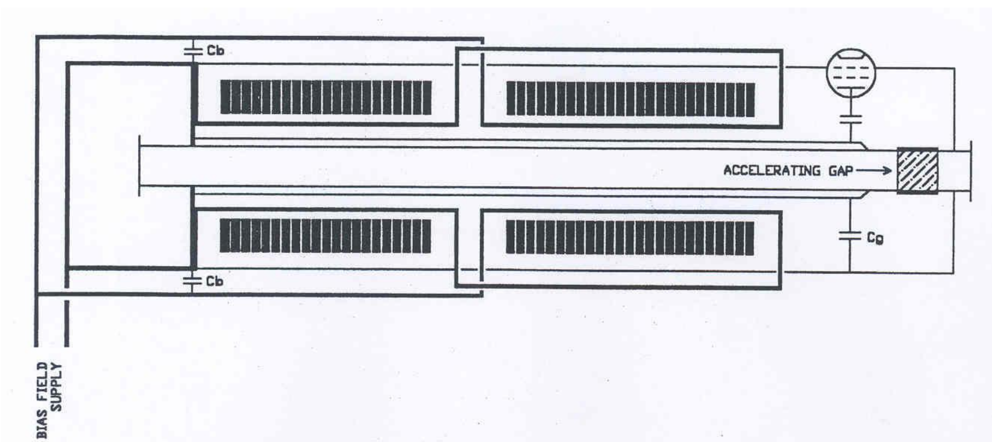
Type $\lambda/4$

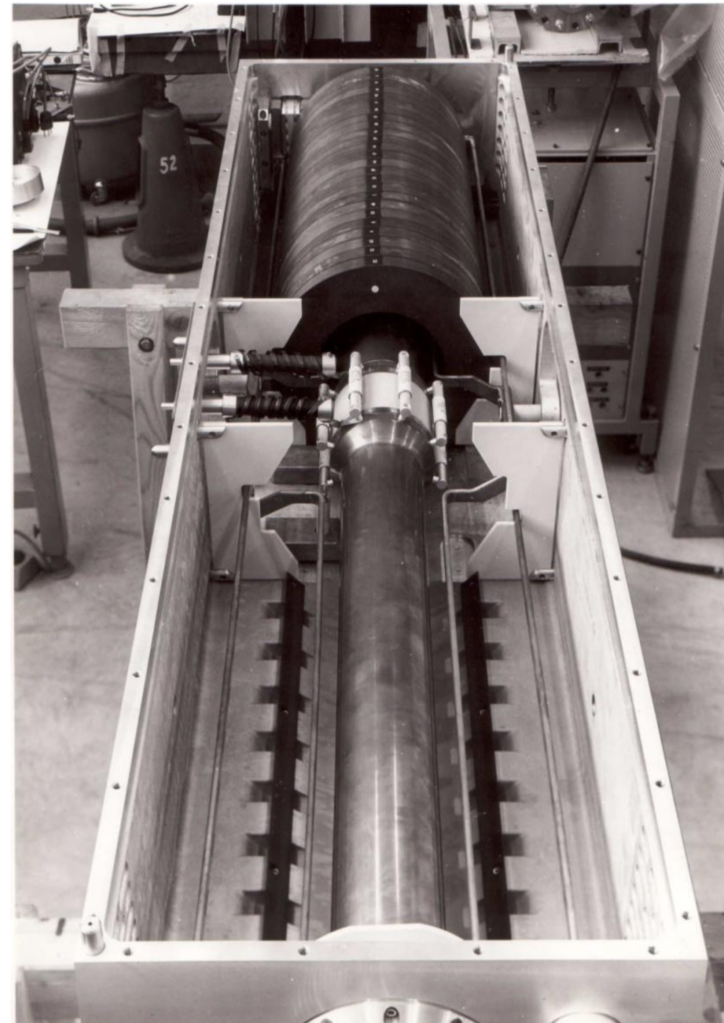


Ferrite Loaded Cavities

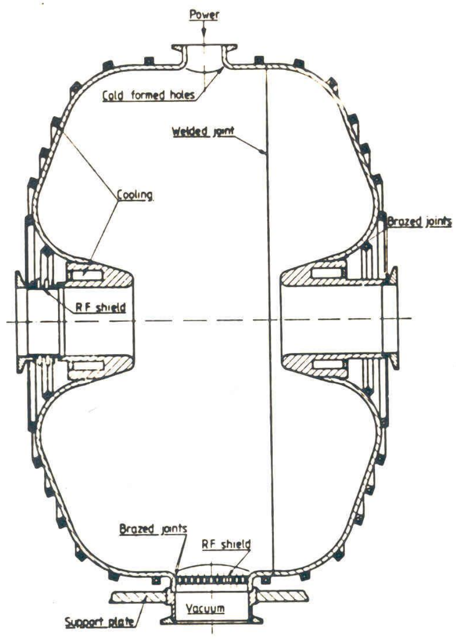
- Ferrite toroids are placed around the beam tube which allow to reach lower frequencies at reasonable size.

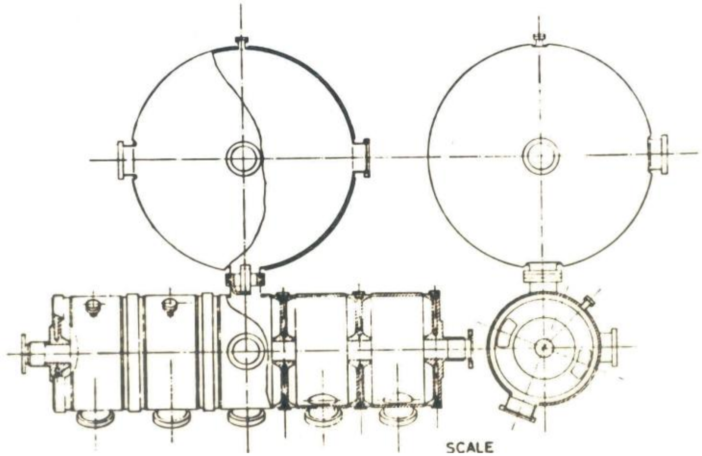
- Polarizing the ferrites will change the resonant frequency, hence satisfying energy ramping in protons and ions synchrotrons.

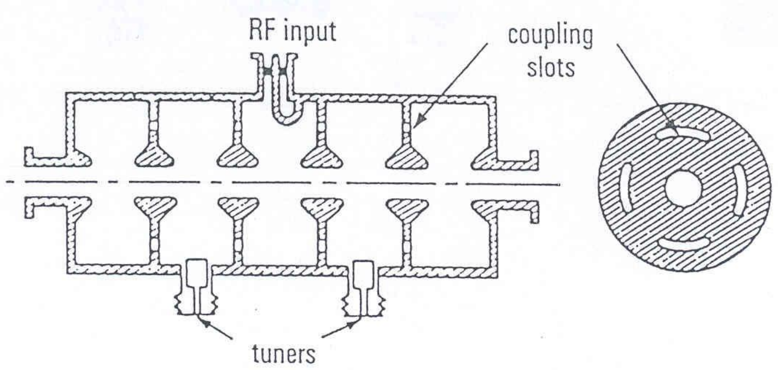


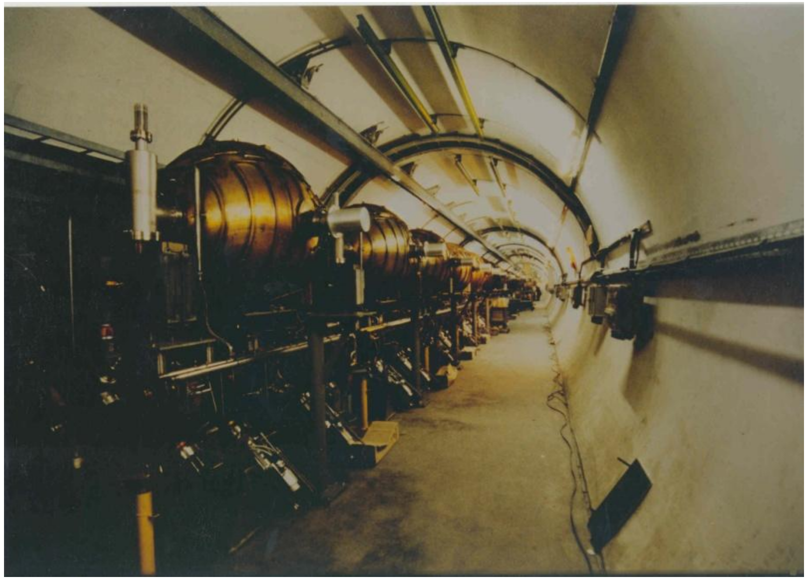


High Q cavities for e⁻ Synchrotrons

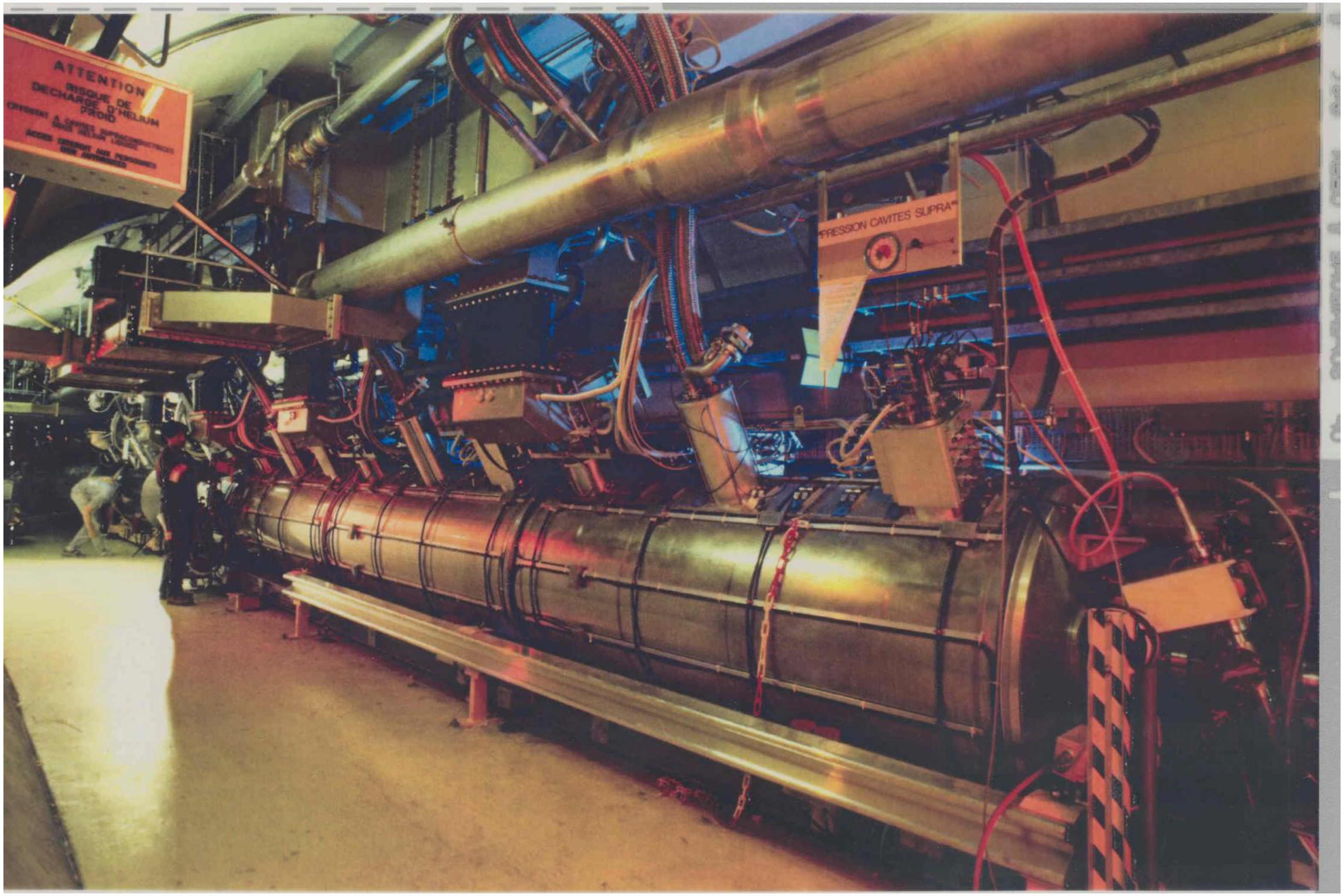




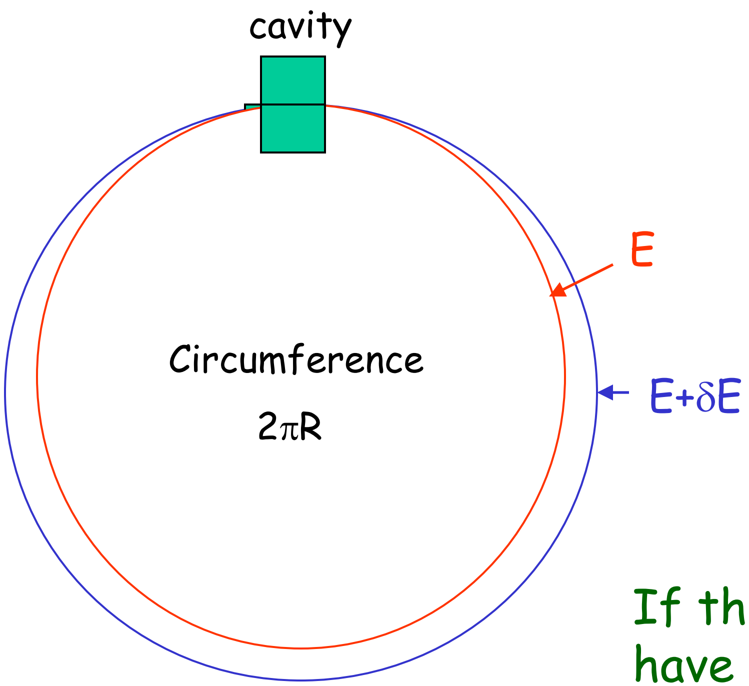




LEP 2: 2x100 GeV with SC cavities



Dispersion Effects in a Synchrotron



If a particle is slightly shifted in momentum it will have a different orbit:

$$\alpha = \frac{p}{R}\frac{dR}{dp}$$

This is the "momentum compaction" generated by the bending field.

If the particle is shifted in momentum it will have also a different velocity. As a result of both effects the revolution frequency changes:

$$\eta = \frac{p}{f_r}\frac{df_r}{dp}$$

p=particle momentum

R=synchrotron physical radius

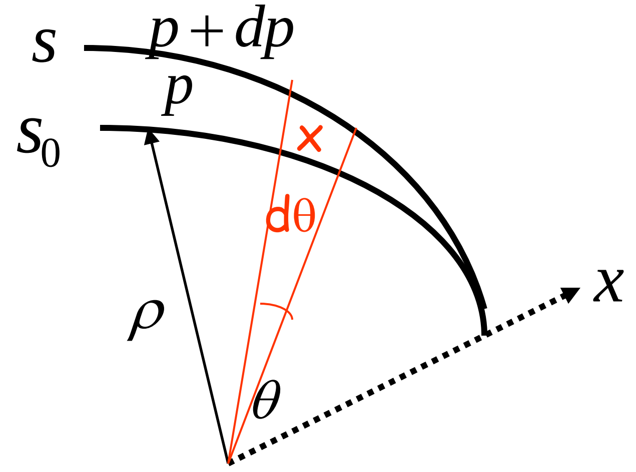
f_r=revolution frequency

Dispersion Effects in a Synchrotron (2)

$$\alpha = \frac{p}{R}\frac{dR}{dp}$$

$$ds_0 = \rho d\theta$$

$$ds = (\rho + x)d\theta$$



The elementary path difference from the two orbits is:

$$\frac{ds - ds_0}{ds_0} = \frac{dl}{ds_0} = \frac{x}{\rho}$$

leading to the total change in the circumference:

$$\int dl = 2\pi dR = \int \frac{x}{\rho} ds_0 = \frac{1}{\rho}\int_m x ds_0 \quad \Rightarrow \quad dR = \langle x \rangle_m$$

$\langle \ \rangle_m$ means that the average is considered over the bending magnet only

Since: $$x = D_x \frac{dp}{p}$$ we get: $$\alpha = \frac{\langle D_x \rangle_m}{R}$$

Dispersion Effects in a Synchrotron (3)

$$\eta = \frac{p}{f_r}\frac{df_r}{dp}$$

$$f_r = \frac{\beta c}{2\pi R} \Rightarrow \frac{df_r}{f_r} = \frac{d\beta}{\beta} - \frac{dR}{R}$$

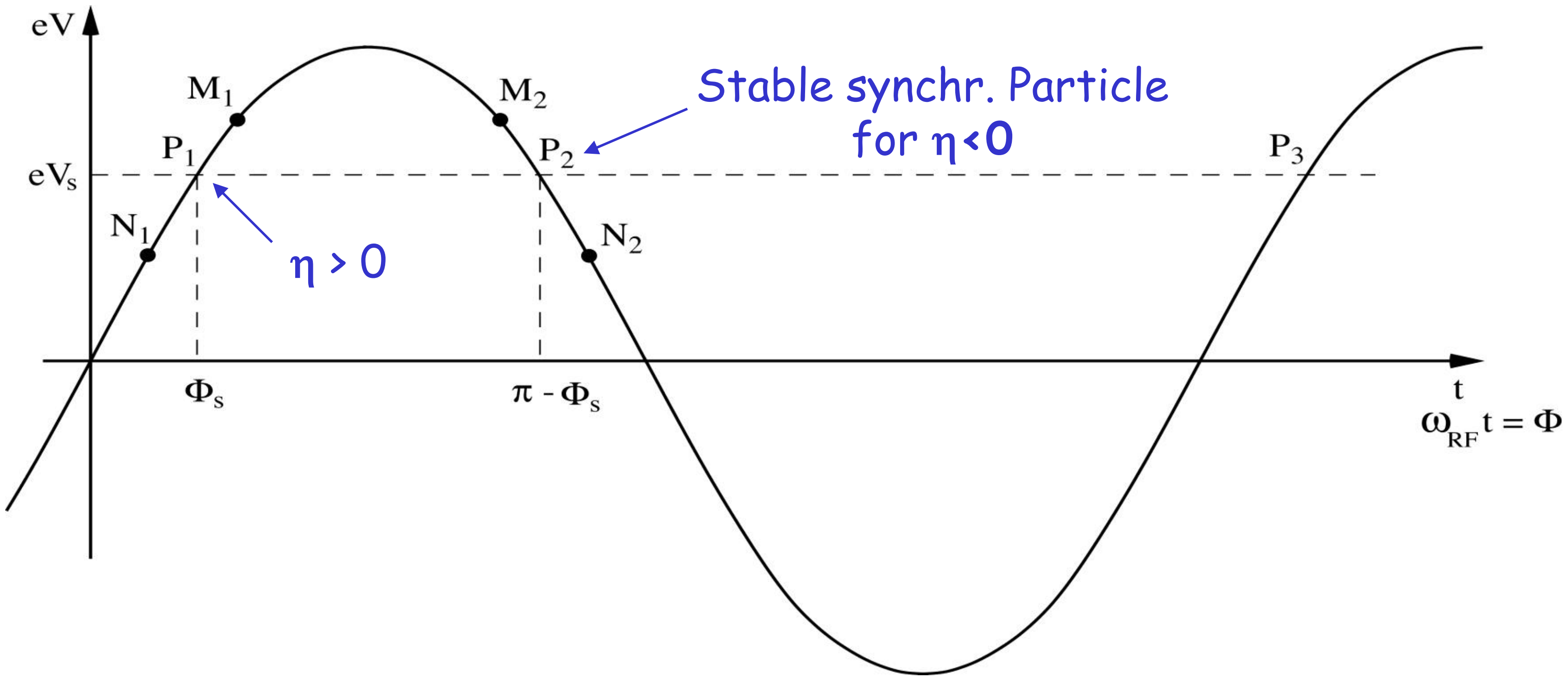
$$p = mv = \beta\gamma\frac{E_0}{c} \Rightarrow \frac{dp}{p} = \frac{d\beta}{\beta} + \frac{d\left(1-\beta^2\right)^{-\frac{1}{2}}}{\left(1-\beta^2\right)^{-\frac{1}{2}}} = \left(1-\beta^2\right)^{-1}\frac{d\beta}{\beta}$$

$$\frac{df_r}{f_r} = \left(\frac{1}{\gamma^2} - \alpha\right)\frac{dp}{p} \longrightarrow \eta = \frac{1}{\gamma^2} - \alpha$$

η=0 at the transition energy $\qquad \gamma_{tr} = \frac{1}{\sqrt{\alpha}}$

Phase Stability in a Synchrotron

From the definition of η it is clear that below transition an increase in energy is followed by a higher revolution frequency (increase in velocity dominates) while the reverse occurs above transition (v ≈ c and longer path) where the momentum compaction (generally > 0) dominates.



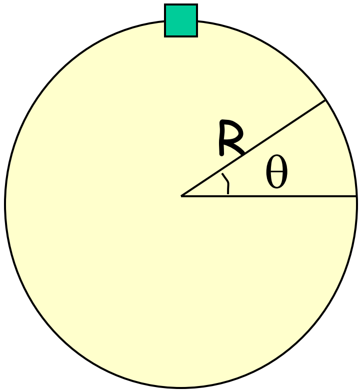
Longitudinal Dynamics

It is also often called " synchrotron motion".

The RF acceleration process clearly emphasizes two coupled variables, the energy gained by the particle and the RF phase experienced by the same particle. Since there is a well defined synchronous particle which has always the same phase ϕ_s, and the nominal energy E_s, it is sufficient to follow other particles with respect to that particle. So let's introduce the following reduced variables:

revolution frequency	:	$\Delta f_r = f_r - f_{rs}$
particle RF phase	:	$\Delta\phi = \phi - \phi_s$
particle momentum	:	$\Delta p = p - p_s$
particle energy	:	$\Delta E = E - E_s$
azimuth angle	:	$\Delta\theta = \theta - \theta_s$

First Energy-Phase Equation



$$f_{RF} = hf_r \quad \Rightarrow \quad \Delta\phi = -h\Delta\theta \quad with \quad \theta = \int \omega_r dt$$

For a given particle with respect to the reference one:

$$\Delta\omega_r = \frac{d}{dt}(\Delta\theta) = -\frac{1}{h}\frac{d}{dt}(\Delta\phi) = -\frac{1}{h}\frac{d\phi}{dt}$$

Since: $\eta = \frac{p_s}{\omega_{rs}}\left(\frac{d\omega_r}{dp}\right)_s$ and $E^2 = E_0^2 + p^2c^2$

$$\Delta E = v_s \Delta p = \omega_{rs} R_s \Delta p$$

one gets:

$$\frac{\Delta E}{\omega_{rs}} = -\frac{p_s R_s}{h\eta\omega_{rs}}\frac{d(\Delta\phi)}{dt} = -\frac{p_s R_s}{h\eta\omega_{rs}}\dot{\phi}$$

Second Energy-Phase Equation

The rate of energy gained by a particle is:

$$\frac{dE}{dt} = e\hat{V}\sin\phi \, \frac{\omega_r}{2\pi}$$

The rate of relative energy gain with respect to the reference particle is then:

$$2\pi\Delta\left(\frac{\dot{E}}{\omega_r}\right) = e\hat{V}(\sin\phi - \sin\phi_s)$$

Expanding the left hand side to first order:

$$\Delta\left(\dot{E}T_r\right) \cong \dot{E}_s \Delta T_r + T_{rs}\Delta\dot{E} = \Delta E\,\dot{T}_{rs} + T_{rs}\Delta\dot{E} = \frac{d}{dt}\left(T_{rs}\Delta E\right)$$

leads to the second energy-phase equation:

$$2\pi\frac{d}{dt}\left(\frac{\Delta E}{\omega_{rs}}\right) = e\hat{V}\left(\sin\phi - \sin\phi_s\right)$$

Equations of Longitudinal Motion

$$\frac{\Delta E}{\omega_{rs}} = -\frac{p_s R_s}{h\eta\omega_{rs}} \frac{d(\Delta\phi)}{dt} = -\frac{p_s R_s}{h\eta\omega_{rs}} \dot{\phi}$$

$$2\pi \frac{d}{dt}\left(\frac{\Delta E}{\omega_{rs}}\right) = e\hat{V}(\sin\phi - \sin\phi_s)$$

deriving and combining

$$\frac{d}{dt}\left[\frac{R_s p_s}{h\eta\omega_{rs}} \frac{d\phi}{dt}\right] + \frac{e\hat{V}}{2\pi}(\sin\phi - \sin\phi_s) = 0$$

This second order equation is non linear. Moreover the parameters within the bracket are in general slowly varying with time…………………

Hamiltonian of Longitudinal Motion

Introducing a new convenient variable, W, leads to the 1th order equations:

$$W = 2\pi\left(\frac{\Delta E}{\omega_{rs}}\right) = 2\pi R_s \Delta p \longrightarrow$$

$$\frac{d\phi}{dt} = -\frac{1}{2\pi}\frac{h\eta\omega_{rs}}{p_s R_s} W$$

$$\frac{dW}{dt} = e\hat{V}\left(\sin\phi - \sin\phi_s\right)$$

These equations of motion derive from a hamiltonian H(ϕ,W,t):

$$\frac{d\phi}{dt} = \frac{\partial H}{\partial W} \qquad \frac{dW}{dt} = -\frac{\partial H}{\partial \phi}$$

$$H(\phi, W, t) = e\hat{V}\left[\cos\phi - \cos\phi_s + (\phi - \phi_s)\sin\phi_s\right] - \frac{1}{4\pi}\frac{h\eta\omega_{rs}}{R_s p_s} W^2$$

Small Amplitude Oscillations

Let's assume constant parameters R_s, p_s, ω_s and η:

$$\ddot{\phi}+\frac{\Omega_s^2}{\cos\phi_s}\left(\sin\phi-\sin\phi_s\right)=0$$

with

$$\Omega_s^2=\frac{h\eta\omega_{rs}e\hat{V}\cos\phi_s}{2\pi R_s p_s}$$

Consider now small phase deviations from the reference particle:

$$\sin\phi-\sin\phi_s=\sin\left(\phi_s+\Delta\phi\right)-\sin\phi_s\cong\cos\phi_s\Delta\phi$$

(for small $\Delta\phi$)

and the corresponding linearized motion reduces to a harmonic oscillation:

$$\ddot{\phi}+\Omega_s^2\Delta\phi=0$$

stable for $\Omega_s^2>0$ and Ω_s real

$\gamma < \gamma_{tr}$	$\eta > 0$	$0 < \phi_s < \pi/2$	$\sin\phi_s > 0$
$\gamma > \gamma_{tr}$	$\eta < 0$	$\pi/2 < \phi_s < \pi$	$\sin\phi_s > 0$

Large Amplitude Oscillations

For larger phase (or energy) deviations from the reference the second order differential equation is non-linear:

$$\ddot{\phi} + \frac{\Omega_s^2}{\cos\phi_s}\left(\sin\phi - \sin\phi_s\right) = 0$$

(Ω_s as previously defined)

Multiplying by $\dot{\phi}$ and integrating gives an invariant of the motion:

$$\frac{\dot{\phi}^2}{2} - \frac{\Omega_s^2}{\cos\phi_s}\left(\cos\phi + \phi\sin\phi_s\right) = I$$

which for small amplitudes reduces to:

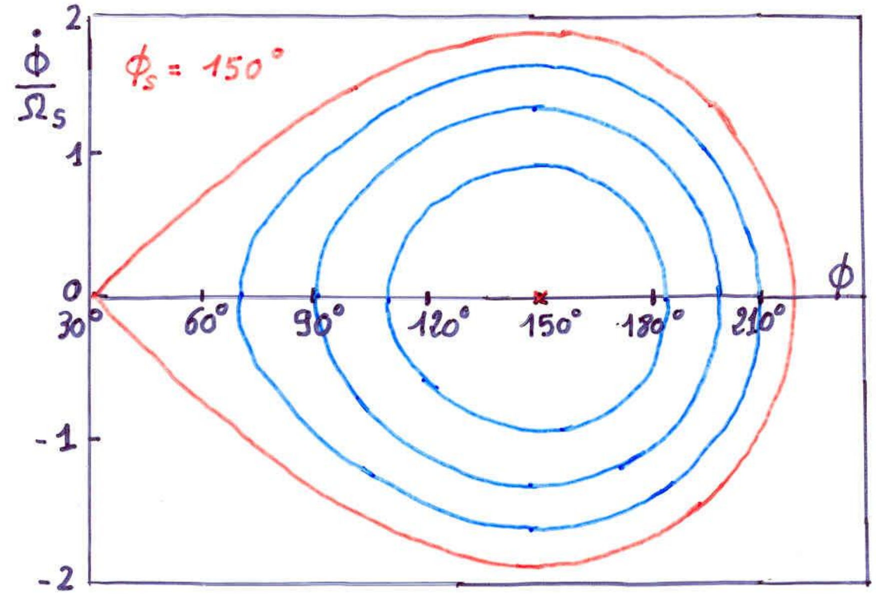
$$\frac{\dot{\phi}^2}{2} + \Omega_s^2\frac{(\Delta\phi)^2}{2} = I$$

(the variable is $\Delta\phi$ and ϕ_s is constant)

Similar equations exist for the second variable : $\Delta E \propto d\phi/dt$

Large Amplitude Oscillations (2)

When ϕ reaches π-ϕ_s the force goes to zero and beyond it becomes non restoring. Hence π-ϕ_s is an extreme amplitude for a stable motion which in the phase space($\frac{\dot{\phi}}{\Omega_s}, \Delta\phi$) is shown as closed trajectories.



Equation of the separatrix:

$$\frac{\dot{\phi}^2}{2} - \frac{\Omega_s^2}{\cos\phi_s}(\cos\phi + \phi\sin\phi_s) = -\frac{\Omega_s^2}{\cos\phi_s}(\cos(\pi-\phi_s) + (\pi-\phi_s)\sin\phi_s)$$

Second value ϕ_m where the separatrix crosses the horizontal axis:

$$\cos\phi_m + \phi_m\sin\phi_s = \cos(\pi-\phi_s) + (\pi-\phi_s)\sin\phi_s$$

Energy Acceptance

From the equation of motion it is seen that $\dot{\phi}$ reaches an extremum when $\ddot{\phi} = 0$, hence corresponding to $\phi = \phi_s$.

Introducing this value into the equation of the separatrix gives:

$$\dot{\phi}_{max}^2 = 2\Omega_s^2 \left\{ 2 + \left(2\phi_s - \pi\right)\tan\phi_s \right\}$$

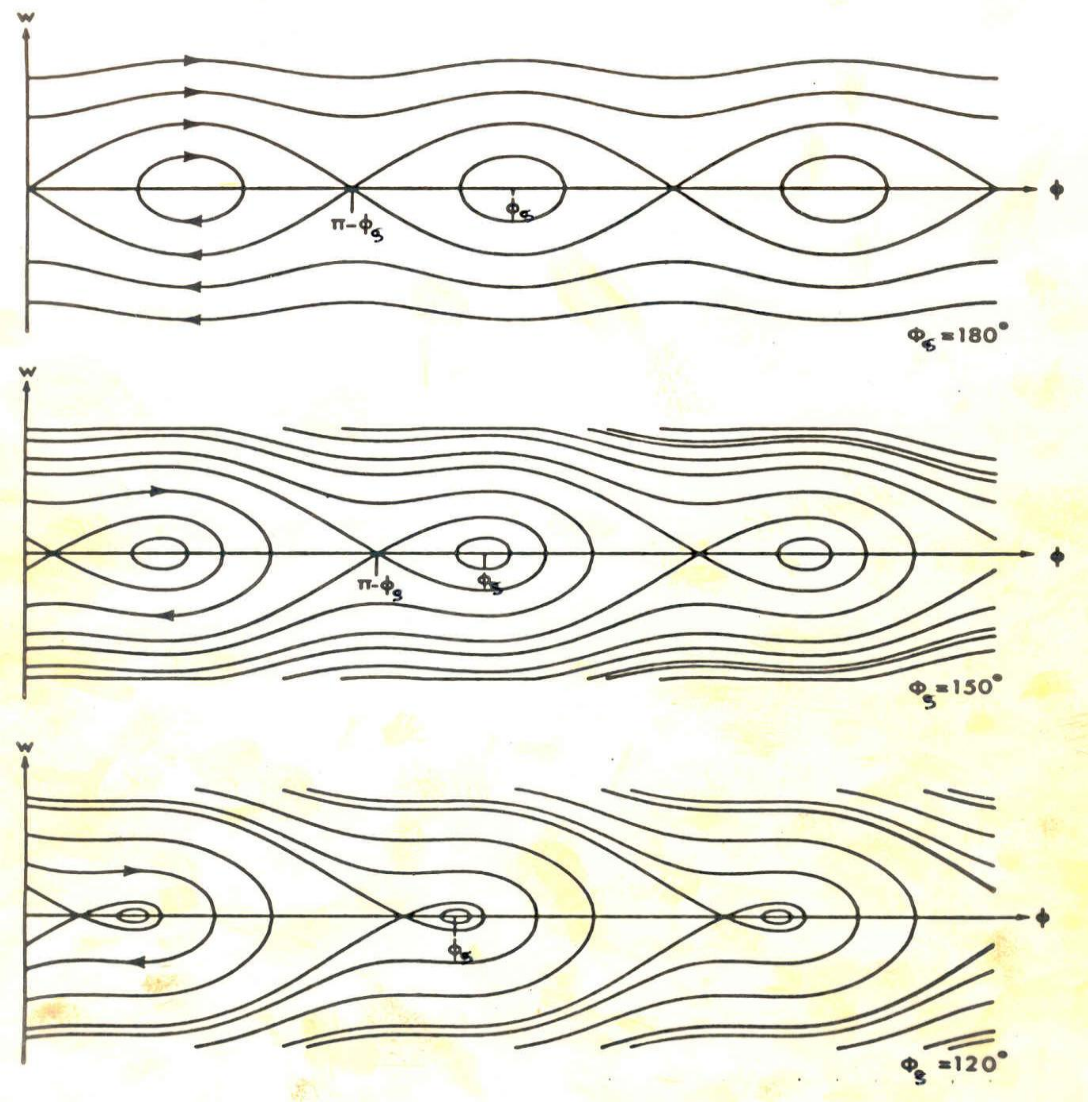
That translates into an acceptance in energy:

$$\left(\frac{\Delta E}{E_s}\right)_{max} = \mp\beta\left\{-\frac{e\hat{V}}{\pi h \eta E_s} G(\phi_s)\right\}^{\frac{1}{2}}$$

$$G(\phi_s) = \left[2\cos\phi_s + (2\phi_s - \pi)\sin\phi_s\right]$$

This "RF acceptance" depends strongly on ϕ_s and plays an important role for the electron capture at injection, and the stored beam lifetime.

RF Acceptance versus Synchronous Phase



As the synchronous phase gets closer to 90° the area of stable motion (closed trajectories) gets smaller. These areas are often called "BUCKET".

The number of circulating buckets is equal to "h".

The phase extension of the bucket is maximum for ϕ_s =180° (or 0°) which correspond to no acceleration . The RF acceptance increases with the RF voltage.

Adiabatic Damping

Though there are many physical processes that can damp the longitudinal oscillation amplitudes, one is directly generated by the acceleration process itself. It will happen in the synchrotron, even ultra-relativistic, when ramping the energy but not in the ultra-relativistic electron linac which does not show any oscillation.

As a matter of fact, when E_s varies with time, one needs to be more careful in combining the two first order energy-phase equations in one second order equation:

$$\frac{d}{dt}\left(E_s\dot{\phi}\right) = -\Omega_s^2 E_s \Delta\phi$$

$$E_s\ddot{\phi} + \dot{E}_s\dot{\phi} + \Omega_s^2 E_s \Delta\phi = 0$$

$$\ddot{\phi} + \frac{\dot{E}_s}{E_s}\dot{\phi} + \Omega_s^2\left(E_s\right)\Delta\phi = 0$$

The damping coefficient is proportional to the rate of energy variation and from the definition of Ω_s one has:

$$\frac{\dot{E}_s}{E_s} = -2\frac{\dot{\Omega}_s}{\Omega_s}$$

Adiabatic Damping (2)

To integrate the previous equation, with variable coefficients, the method consists of choosing a solution similar to the one obtained without the damping term:

$$\Delta\phi = \Delta\hat{\phi}(t)\sin\left[\int^{t}\Omega_s(\tau)d\tau + const.\right] = \Delta\hat{\phi}(t)\sin\psi(t)$$

Assuming time derivatives of parameters are small quantities (adiabatic limit), putting the solution in the equation and neglecting second order terms one gets:

$$\left[2\Delta\dot{\hat{\phi}}\Omega_s - \Delta\hat{\phi}\dot{\Omega}_s\right]\cos\psi(t) = 0$$

$$2\frac{\Delta\dot{\hat{\phi}}}{\Delta\hat{\phi}} - \frac{\dot{\Omega}_s}{\Omega_s} = 0$$

Integrating:

$$\Delta\hat{\phi} \propto \Omega_s^{1/2} \qquad\qquad \Delta\hat{\phi} \propto E_s^{-1/4}$$

Adiabatic Damping (3)

So far it was assumed that parameters related to the acceleration process were constant. Let's consider now that they vary slowly with respect to the period of longitudinal oscillation (adiabaticity).

For small amplitude oscillations the hamiltonian reduces to:

$$H(\phi,W,t) \cong -\frac{e\hat{V}}{2}\cos\phi_s(\Delta\phi)^2 - \frac{1}{4\pi}\frac{h\eta\omega_{rs}}{R_s p_s}W^2$$

with

$$W = \hat{W}\cos\Omega_s t$$

$$\Delta\phi = (\Delta\hat{\phi})\sin\Omega_s t$$

Under adiabatic conditions the Boltzman-Ehrenfest theorem states that the action integral remains constant:

$$I = \oint W\,d\phi = const.$$

(W, ϕ are canonical variables)

Since:

$$\frac{d\phi}{dt} = \frac{\partial H}{\partial W} = -\frac{1}{2\pi}\frac{h\eta\omega_{rs}}{R_s p_s}W$$

the action integral becomes:

$$I = \oint W\frac{d\phi}{dt}dt = -\frac{1}{2\pi}\frac{h\eta\omega_{rs}}{R_s p_s}\oint W^2 dt$$

Adiabatic Damping (4)

Previous integral over one period:

$$\oint W^2 dt = \pi \frac{\hat{W}^2}{\Omega_s}$$

leads to:

$$I = -\frac{h\eta\omega_{rs}}{2R_s p_s}\frac{\hat{W}^2}{\Omega_s} = const.$$

From the quadratic form of the hamiltonian one gets the relation:

$$\hat{W} = \frac{2\pi p_s R_s \Omega_s}{h\eta\omega_{rs}} \Delta\hat{\phi}$$

Finally under adiabatic conditions the long term evolution of the oscillation amplitudes is shown to be:

$$\Delta\hat{\phi} \propto \left[\frac{\eta}{E_s R_s^2 \hat{V} \cos\phi_s}\right]^{1/4} \propto E_s^{-1/4}$$

$$\hat{W} \; or \; \Delta\hat{E} \propto E_s^{1/4}$$

$$\hat{W}.\Delta\hat{\phi} = invariant$$

Dynamics in the Vicinity of Transition Energy

Introducing in the previous expressions: $\eta = \frac{1}{\gamma^2} - \alpha = \gamma^{-2} - \gamma_t^{-2}$

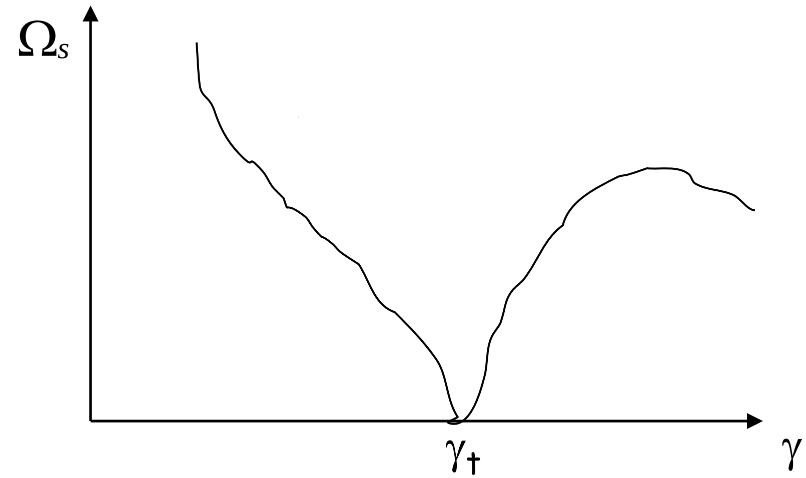
one gets:

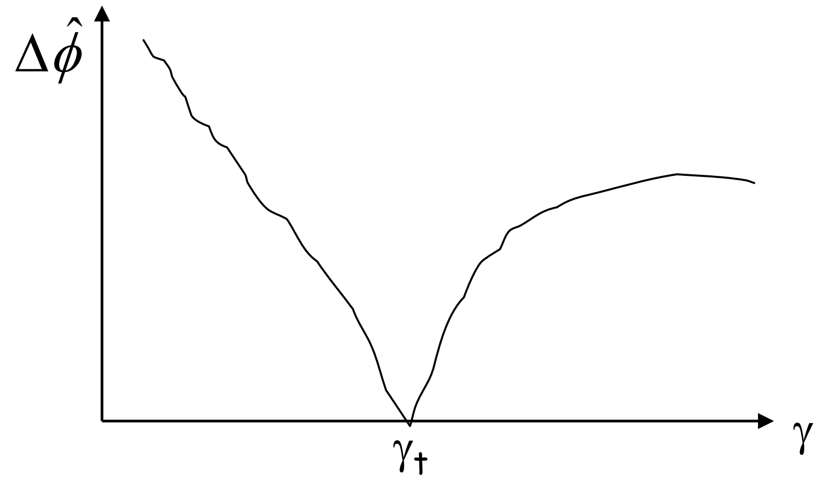
$$\Delta\hat{\phi} \propto \left\{ \frac{1}{\hat{V}\left|\cos\phi_s\right|} \frac{\left|\gamma^{-2} - \gamma_t^{-2}\right|}{\gamma} \right\}^{1/4}$$

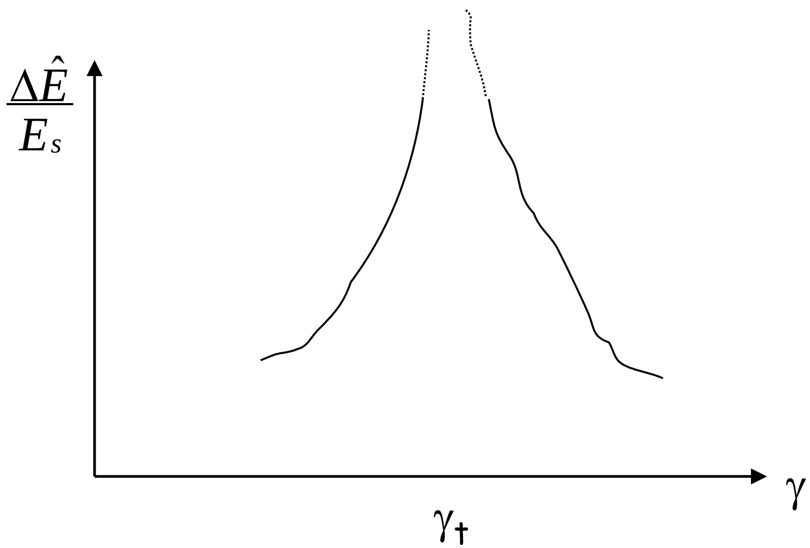
$$\Delta\hat{E} \propto \left\{ \frac{1}{\hat{V}\left|\cos\phi_s\right|} \frac{\left|\gamma^{-2} - \gamma_t^{-2}\right|}{\gamma} \right\}^{-1/4}$$

$$\Omega_s \propto \left\{ \hat{V}\left|\cos\phi_s\right| \frac{\left|\gamma^{-2} - \gamma_t^{-2}\right|}{\gamma} \right\}^{1/2}$$

Dynamics in the Vicinity of Transition Energy (2)







In fact close to transition, adiabatic solution are not valid since parameters change too fast. A proper treatment would show that:

$\Delta\phi$ will not go to zero

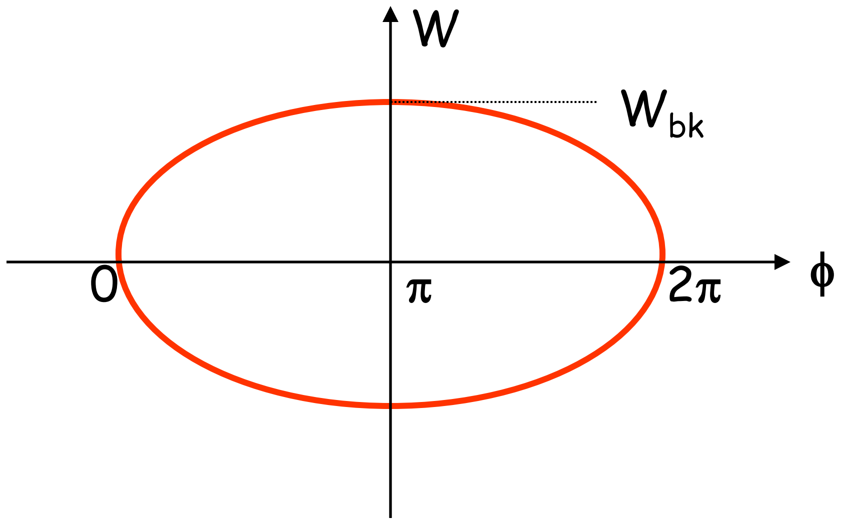
ΔE will not go to infinity

Stationary Bucket

This is the case $\sin\phi_s=0$ (no acceleration) which means $\phi_s=0$ or π . The equation of the separatrix for $\phi_s=\pi$ (above transition) becomes:

$$\frac{\dot{\phi}^2}{2}+\Omega_s^2\cos\phi=\Omega_s^2 \qquad \frac{\dot{\phi}^2}{2}=2\Omega_s^2\sin^2\frac{\phi}{2}$$

Replacing the phase derivative by the canonical variable W:



$$W=2\pi\frac{\Delta E}{\omega_{rs}}=-2\pi\frac{p_s R_s}{h\eta\omega_{rs}}\dot{\phi}$$

and introducing the expression for Ω_s leads to the following equation for the separatrix:

$$W=\pm 2\frac{C}{c}\sqrt{\frac{-e\hat{V}E_s}{2\pi h\eta}}\sin\frac{\phi}{2}$$

with $C=2\pi R_s$

Stationary Bucket (2)

Setting $\phi=\pi$ in the previous equation gives the height of the bucket:

$$W_{bk} = 2\frac{C}{c}\sqrt{\frac{-e\hat{V}E_s}{2\pi h\eta}}$$

The area of the bucket is:

$$A_{bk} = 2\int_0^{2\pi} W\, d\phi$$

Since:

$$\int_0^{2\pi} \sin\frac{\phi}{2}\, d\phi = 4$$

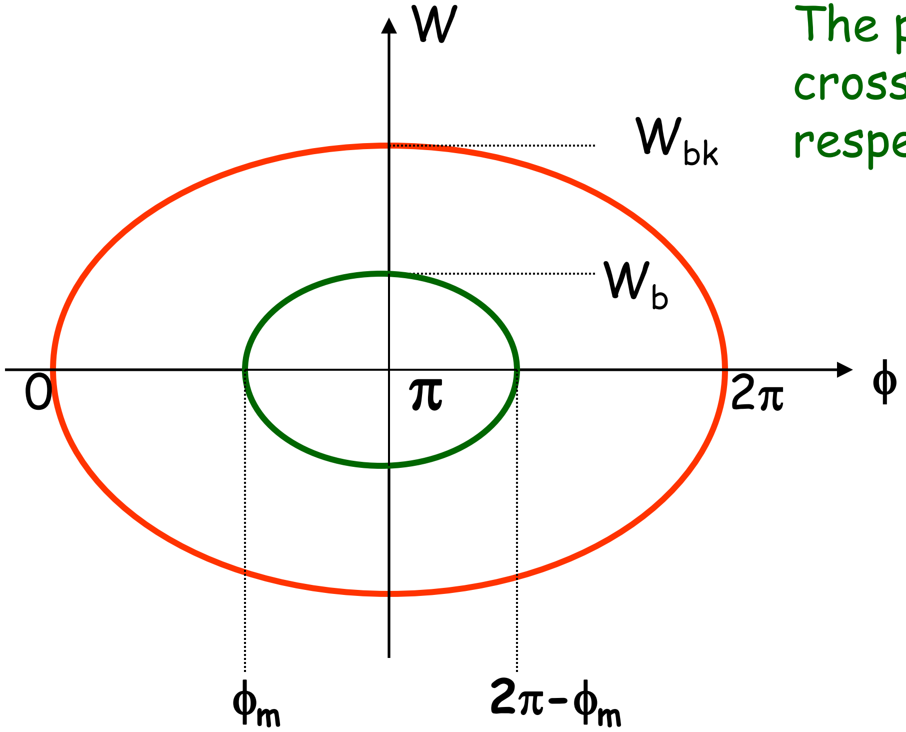
one gets:

$$A_{bk} = 16\frac{C}{c}\sqrt{\frac{-e\hat{V}E_s}{2\pi h\eta}} \longrightarrow W_{bk} = \frac{A_{bk}}{8}$$

Bunch Matching into a Stationary Bucket

A particle trajectory inside the separatrix is described by the equation:

$$\frac{\dot{\phi}^2}{2} - \frac{\Omega_s^2}{\cos\phi_s}\left(\cos\phi + \phi\sin\phi_s\right) = I \quad \xrightarrow{\phi_s = \pi} \quad \frac{\dot{\phi}^2}{2} + \Omega_s^2\cos\phi = I$$



The points where the trajectory crosses the axis are symmetric with respect to $\phi_s = \pi$

$$\frac{\dot{\phi}^2}{2} + \Omega_s^2\cos\phi = \Omega_s^2\cos\phi_m$$

$$\dot{\phi} = \pm\Omega_s\sqrt{2\left(\cos\phi_m - \cos\phi\right)}$$

$$W = \pm\frac{A_{bk}}{8}\sqrt{\cos^2\frac{\phi_m}{2} - \cos^2\frac{\phi}{2}}$$

Bunch Matching into a Stationary Bucket (2)

Setting $\phi = \pi$ in the previous formula allows to calculate the bunch height:

$$W_b = \frac{A_{bk}}{8}\cos\frac{\phi_m}{2}$$

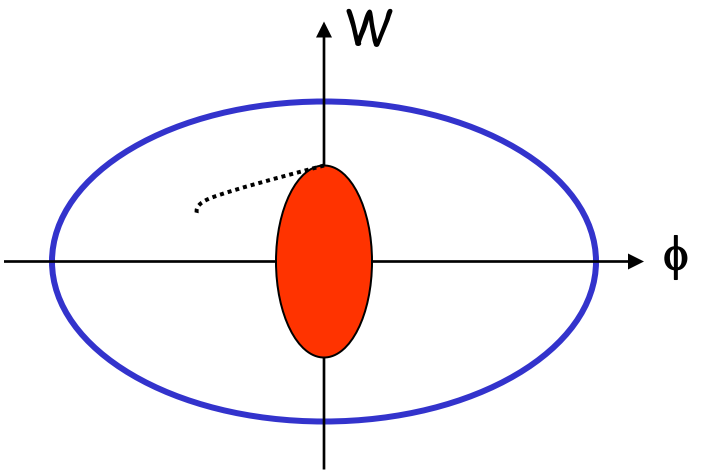
or:

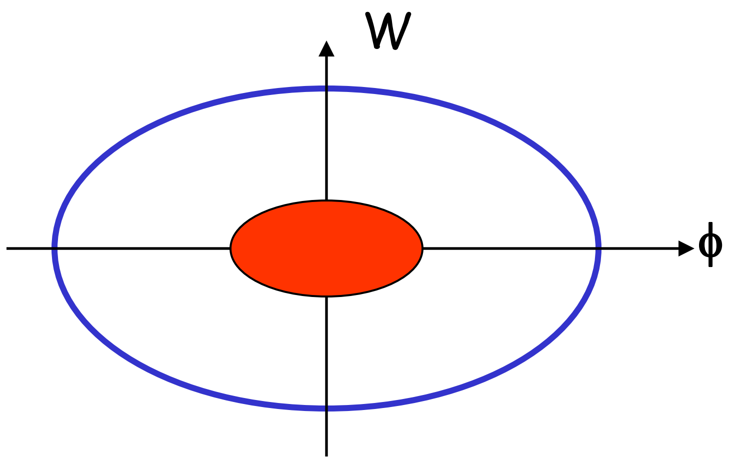
$$W_b = W_{bk}\cos\frac{\phi_m}{2} \longrightarrow \left(\frac{\Delta E}{E_s}\right)_b = \left(\frac{\Delta E}{E_s}\right)_{RF}\cos\frac{\phi_m}{2}$$

This formula shows that for a given bunch energy spread the proper matching of a shorter bunch will require a bigger RF acceptance, hence a higher voltage (short bunch means ϕ_m close to π).

Effect of a Mismatch

Starting with an injected bunch with short lenght and large energy spread, after a quarter of synchrotron period the bunch rotation shows a longer bunch with a smaller energy spread.





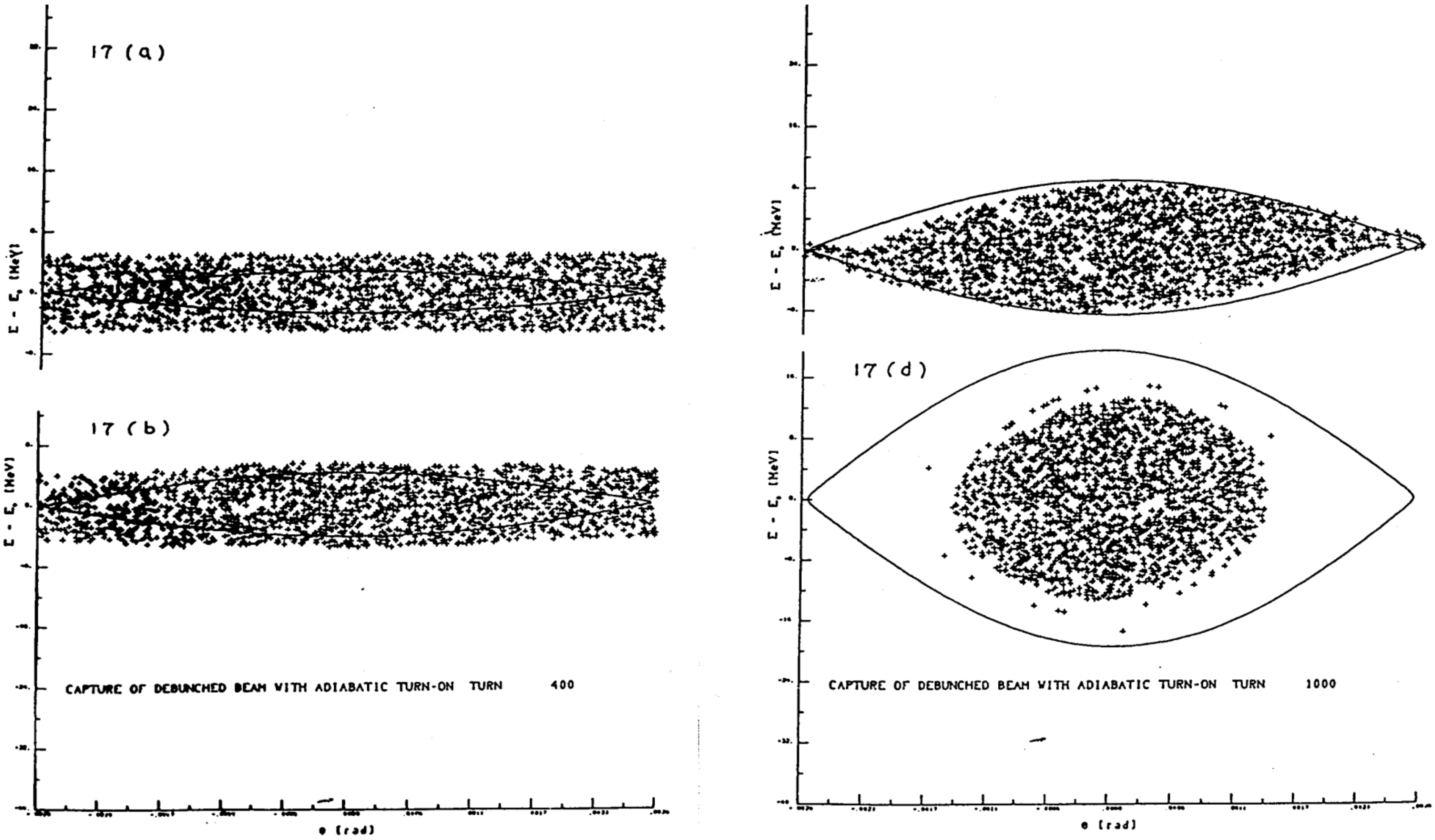
For small oscillation amplitudes the equation of the ellipse reduces to:

$$W = \frac{A_{bk}}{16}\sqrt{(\Delta\phi)_m^2 - (\Delta\phi)^2} \longrightarrow \left(\frac{16W}{A_{bk}(\Delta\phi)_m}\right)^2 + \left(\frac{\Delta\phi}{(\Delta\phi)_m}\right)^2 = 1$$

Ellipse area is called longitudinal emittance

$$A_b = \frac{\pi}{16} A_{bk} (\Delta\phi)_m^2$$

Capture of a Debunched Beam with Adiabatic Turn-On



Capture of a Debunched Beam with Fast Turn-On