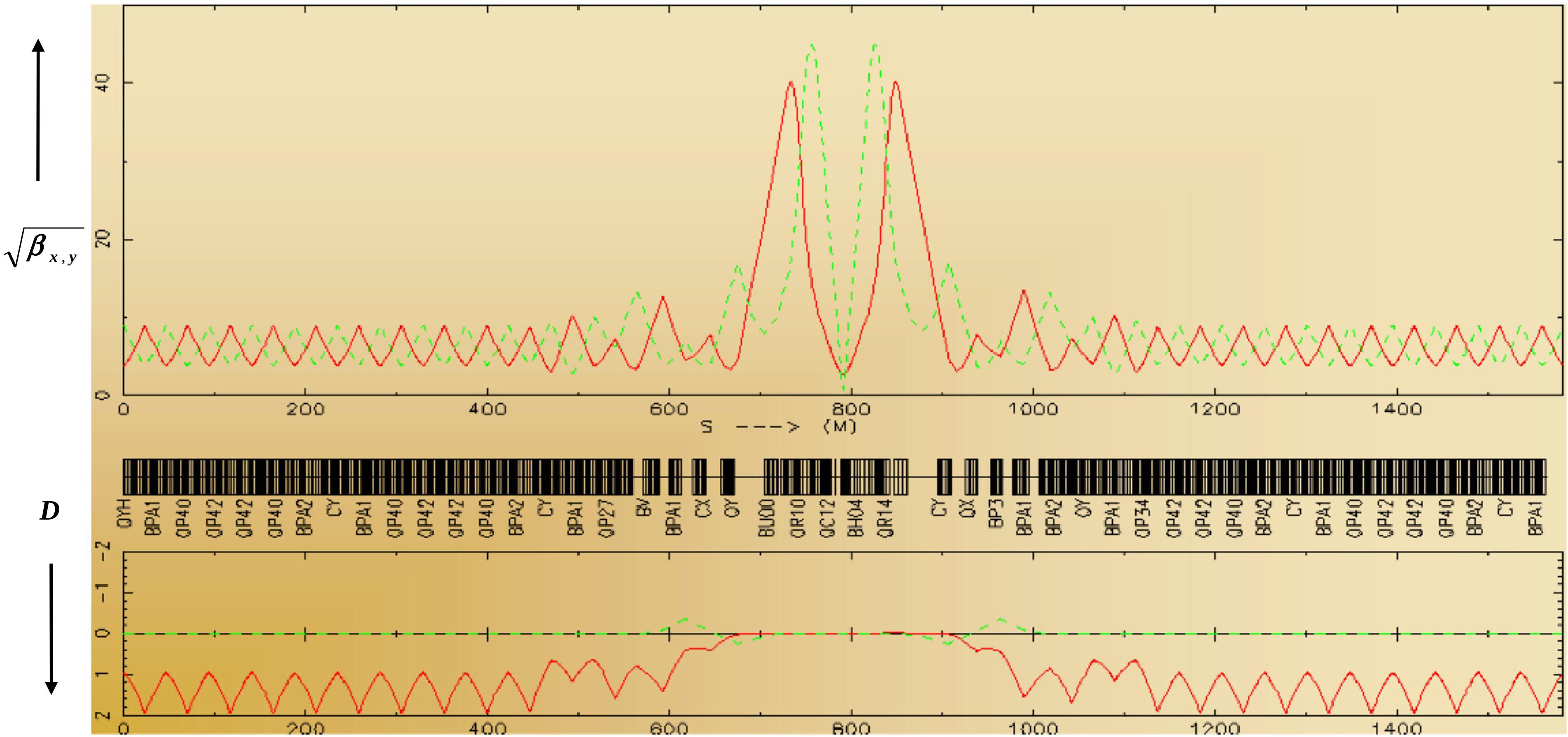
Introduction to Transverse Beam Optics

Bernhard Holzer, CERN

III.) Twiss Parameters & Lattice Design

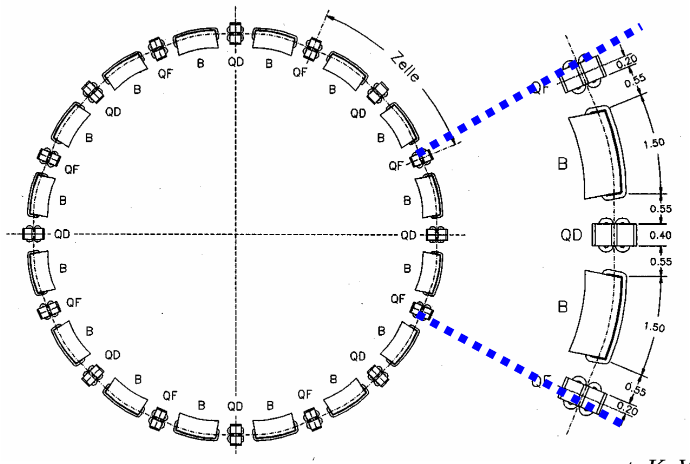
The „ not so ideal world “



Reminder: Particle Trajectories

Transformation through a lattice:

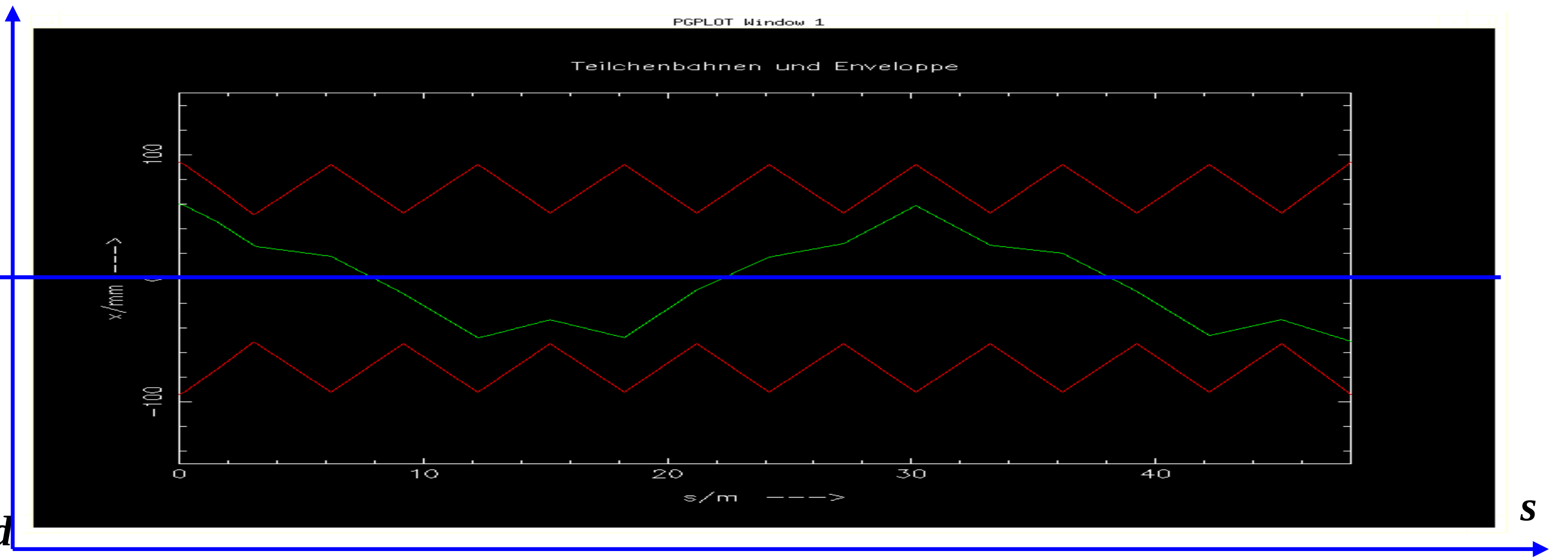
combine the single element solutions by multiplication of the matrices



court. K. Wille

$$M_{total} = M_{QF} * M_{D} * M_{QD} * M_{Bend} * M_{D*.....}$$

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s2} = M(s2,s1) * \begin{pmatrix} x \\ x' \end{pmatrix}_{s1}$$



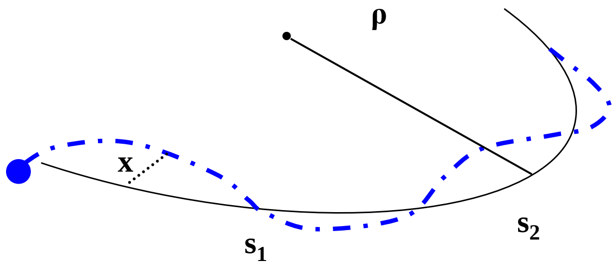
typical values in a strong foc. machine:

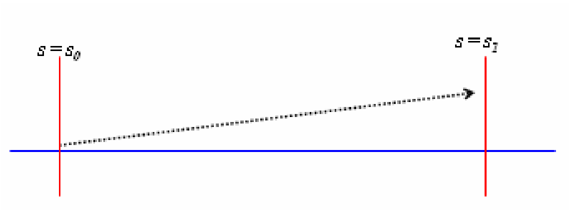
x ≈ mm, x´ ≤ mrad

Equation of Motion:

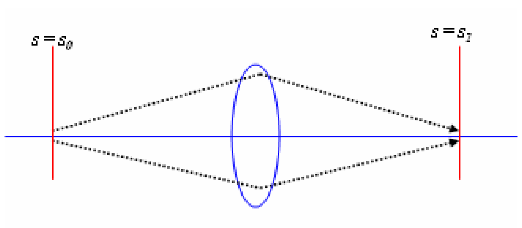
$$x'' + \boldsymbol{K}\ x = 0 \qquad K = 1/\rho^2 - k \quad \textbf{\textit{... hor. plane:}}$$

$$K = k \quad \textbf{\textit{... vert. Plane:}}$$

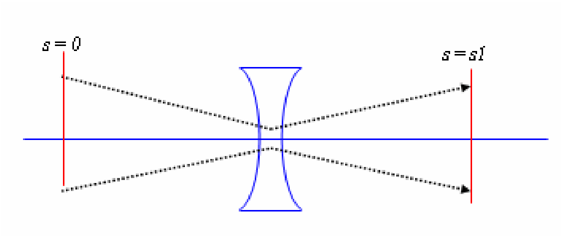




$$\boldsymbol{M}_{drift} = \begin{pmatrix} 1 & \boldsymbol{l} \\ 0 & 1 \end{pmatrix}$$



$$\boldsymbol{M}_{foc} = \begin{pmatrix} \cos(\sqrt{|\boldsymbol{K}|}\boldsymbol{l}) & \frac{1}{\sqrt{|\boldsymbol{K}|}}\sin(\sqrt{|\boldsymbol{K}|}\boldsymbol{l}) \\ -\sqrt{|\boldsymbol{K}|}\sin(\sqrt{|\boldsymbol{K}|}\boldsymbol{l}) & \cos(\sqrt{|\boldsymbol{K}|}\boldsymbol{l}) \end{pmatrix}$$

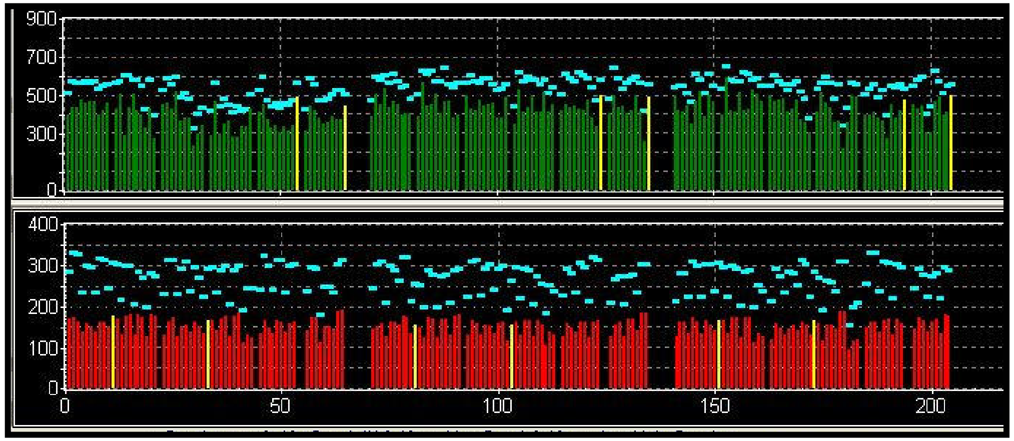


$$\boldsymbol{M}_{defoc} = \begin{pmatrix} \cosh(\sqrt{|\boldsymbol{K}|}\boldsymbol{l} & \frac{1}{\sqrt{|\boldsymbol{K}|}}\sinh(\sqrt{|\boldsymbol{K}|}\boldsymbol{l} \\ \sqrt{|\boldsymbol{K}|}\sinh(\sqrt{|\boldsymbol{K}|}\boldsymbol{l} & \cosh(\sqrt{|\boldsymbol{K}|}\boldsymbol{l} \end{pmatrix}$$

Reminder: β function & Beam Emittance ε

Hill equation

$$x''(s) + \left\{ \frac{1}{\rho^2(s)} - k(s) \right\} * x(s) = 0$$



HERA number of stored particles:

$$N_b = \frac{100mA}{180} * \frac{\tau_{rev}}{e}$$

$$N_b = \frac{100 * 10^{-3}}{180} * \frac{Cb}{s} * \frac{21 * 10^{-6}}{1.6 * 10^{-19}} * \frac{s}{Cb}$$

$$N_b = 7.3 * 10^{10}$$

General solution of Hill´s equation:

$$x(s) = \sqrt{\varepsilon}\sqrt{\beta(s)} \cdot \cos(\psi(s) + \phi)$$

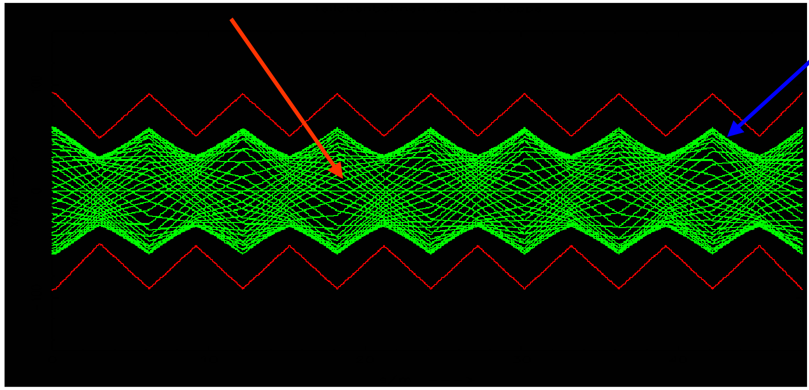
ε = constant

β(s) = periodic function given by focusing properties

$$\beta(s+L) = \beta(s)$$

Emittance of the Particle Ensemble:

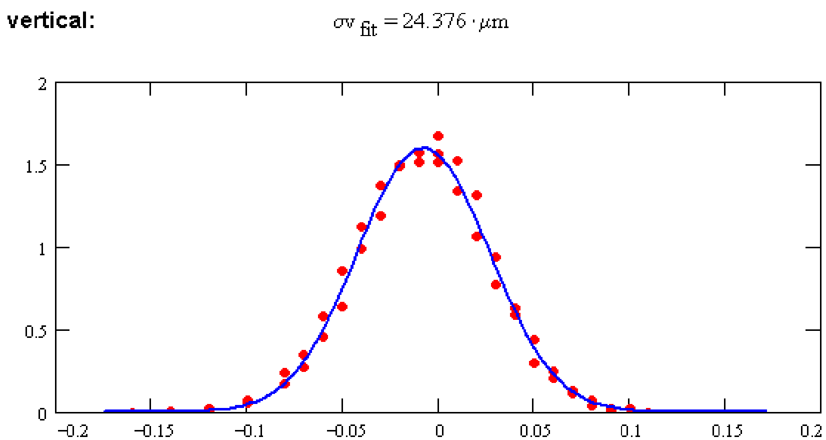
$$x(s) = \sqrt{\varepsilon}\sqrt{\beta(s)} \cdot \cos(\Psi(s) + \phi) \qquad \hat{x}(s) = \sqrt{\varepsilon}\sqrt{\beta(s)}$$

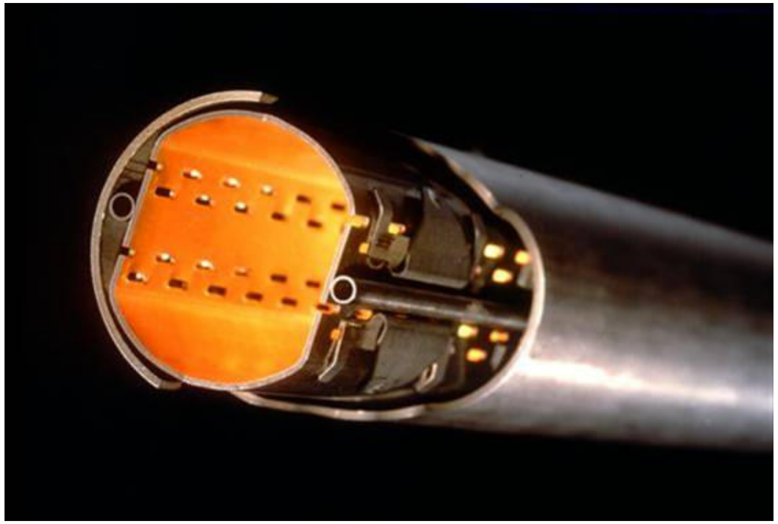


single particle trajectories, N ≈ 10 ^{11} per bunch

Gauß
Particle Distribution: $\rho(x) = \dfrac{N \cdot e}{\sqrt{2\pi}\sigma_x} \cdot e^{-\frac{1}{2}\frac{x^2}{\sigma_x{}^2}}$

particle at distance 1 σ from centre ↔ 68.3 % of all beam particles



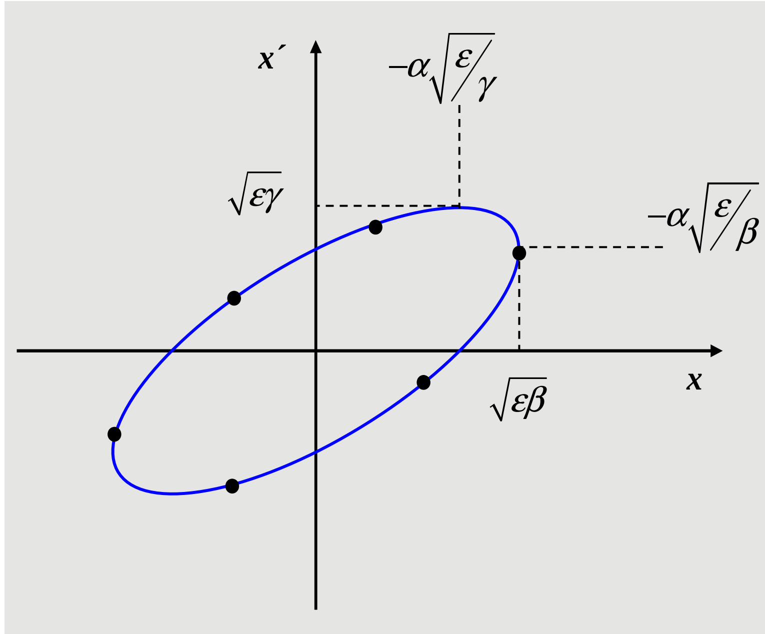


LHC: $\sigma = \sqrt{\varepsilon * \beta} = \sqrt{5*10^{-10}\,m*180\,m} = 0.3\,mm$

aperture requirements: r_0 = 10 * σ

Phase Space Ellipse

$$\varepsilon = \gamma(s)\, x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s)\, x'^2(s)$$



shape and orientation of the phase space ellipse depend on the Twiss parameters β α γ

Calculation of the Twiss Parameters

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s2} = M\,(\mathrm{s2,s1}) * \begin{pmatrix} x \\ x' \end{pmatrix}_{s1}$$

$$M_{total} = M_{QF} * M_{D} * M_{QD} * M_{Bend} * M_{D*.....}$$

$$M(s) = \begin{pmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos(\mu) - \alpha\sin\mu \end{pmatrix} \rightarrow$$

β

α

μ

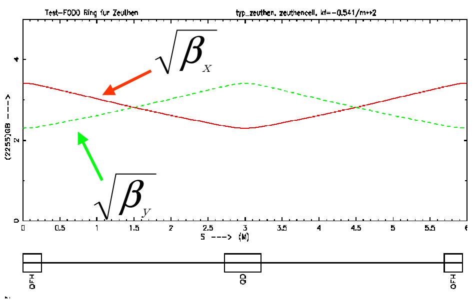
Transformation Matrix & Twiss Parameters:

***transfer matrix** for particle trajectories*
as a function of the lattice parameters

$$M_{1\to 2}=\begin{pmatrix}\sqrt{\frac{\beta_2}{\beta_1}}(\cos\psi_{12}+\alpha_1\sin\psi_{12}) & \sqrt{\beta_1\beta_2}\sin\psi_{12}\\ \frac{(\alpha_1-\alpha_2)\cos\psi_{12}-(1+\alpha_1\alpha_2)\sin\psi_{12}}{\sqrt{\beta_1\beta_2}} & \sqrt{\frac{\beta_1}{\beta_2}}(\cos\psi_{12}-\alpha_2\sin\psi_{12})\end{pmatrix}$$

transfer matrix for periodic structures
... to determine the Twiss babies

$$M(s)=\begin{pmatrix}\cos\psi_{turn}+\alpha_s\sin\psi_{turn} & \beta_s\sin\psi_{turn}\\ -\gamma_s\sin\psi_{turn} & \cos\psi_{turn}-\alpha_s\sin\psi_{turn}\end{pmatrix}$$



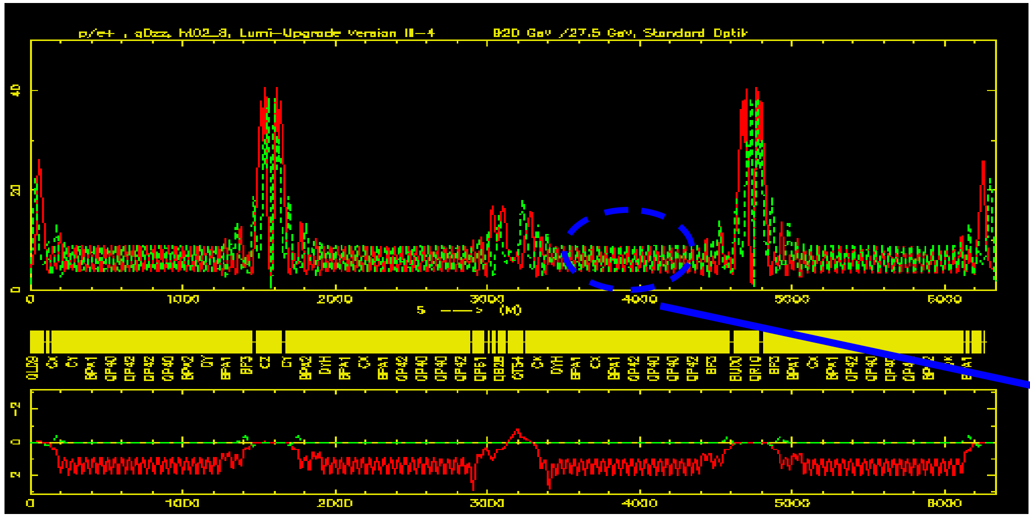
elementary cell of a storage ring

transfer matrix for Twiss parameters
between two lattice locations

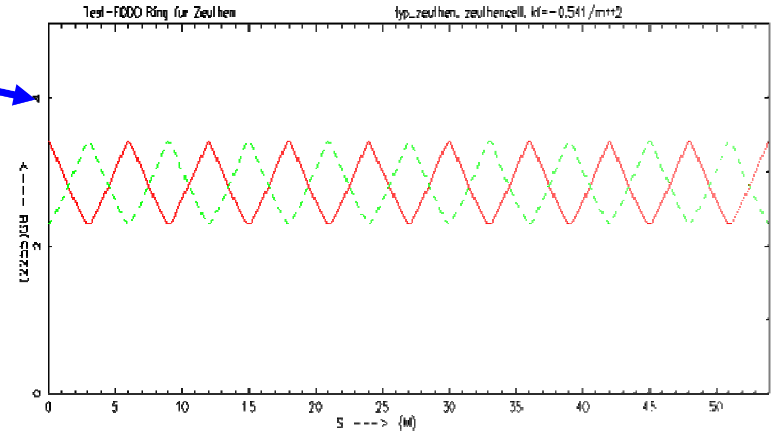
$$\begin{pmatrix}\beta\\ \alpha\\ \gamma\end{pmatrix}_s=\begin{pmatrix}C^2 & -2SC & S^2\\ -CC' & SC'+CS' & -SS'\\ C'^2 & -2S'C' & S'^2\end{pmatrix}\cdot\begin{pmatrix}\beta_0\\ \alpha_0\\ \gamma_0\end{pmatrix}$$

$$M_{total}=\begin{pmatrix}C & S\\ C' & S'\end{pmatrix}=M_{QF}*M_{D1}*M_{Bend}*M_{D2}*M_{QD}\ \ldots$$

Layout of a storage ring lattice



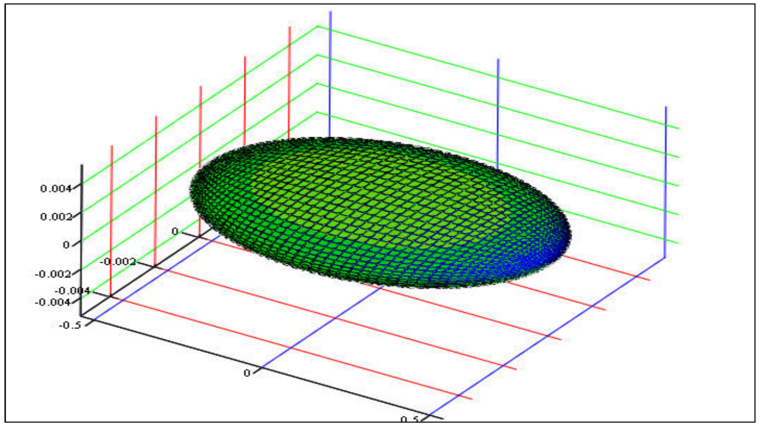
calculate the periodic solution



determine the beta functions in the cell

$$\hat{\beta} = \frac{(1+\sin\frac{\mu}{2})L}{\sin\mu}$$

$$\check{\beta} = \frac{(1-\sin\frac{\mu}{2})L}{\sin\mu}$$



(Z , X , Y)

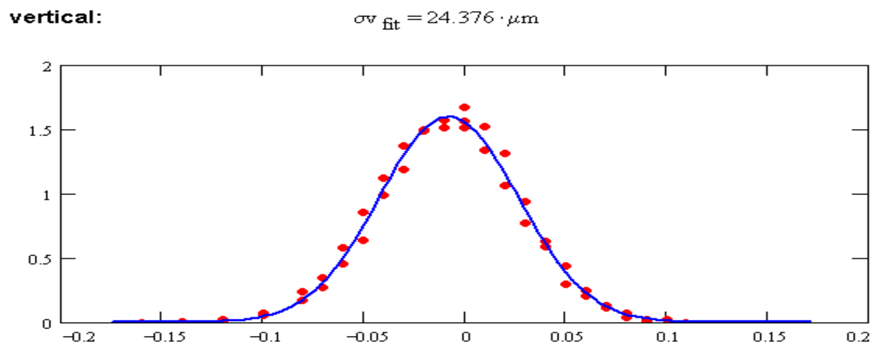
to get the size of a particle bunch

15.) Beam dimension:

Optimisation of the FoDo Phase advance:

In both planes a gaussian particle distribution is assumed, given by the beam emittance ε and the β-function

$$\sigma = \sqrt{\varepsilon\beta}$$



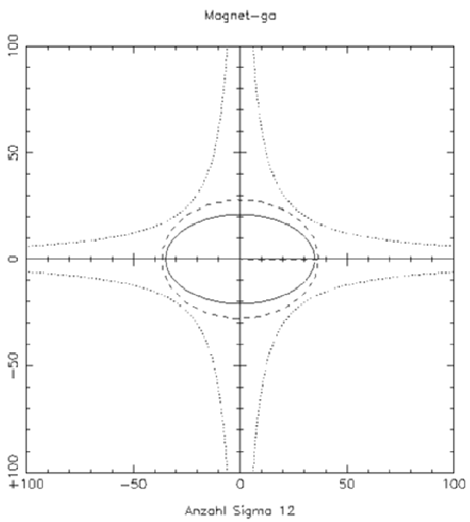
HERA beam size

In general proton beams are „round“ in the sense that

$$\varepsilon_x \approx \varepsilon_y$$

So for highest aperture we have to minimise the β-function in both planes:

$$r^2 = \varepsilon_x\beta_x + \varepsilon_y\beta_y$$



typical beam envelope, vacuum chamber and pole shape in a foc. Quadrupole lens in HERA

Optimising the FoDo phase advance

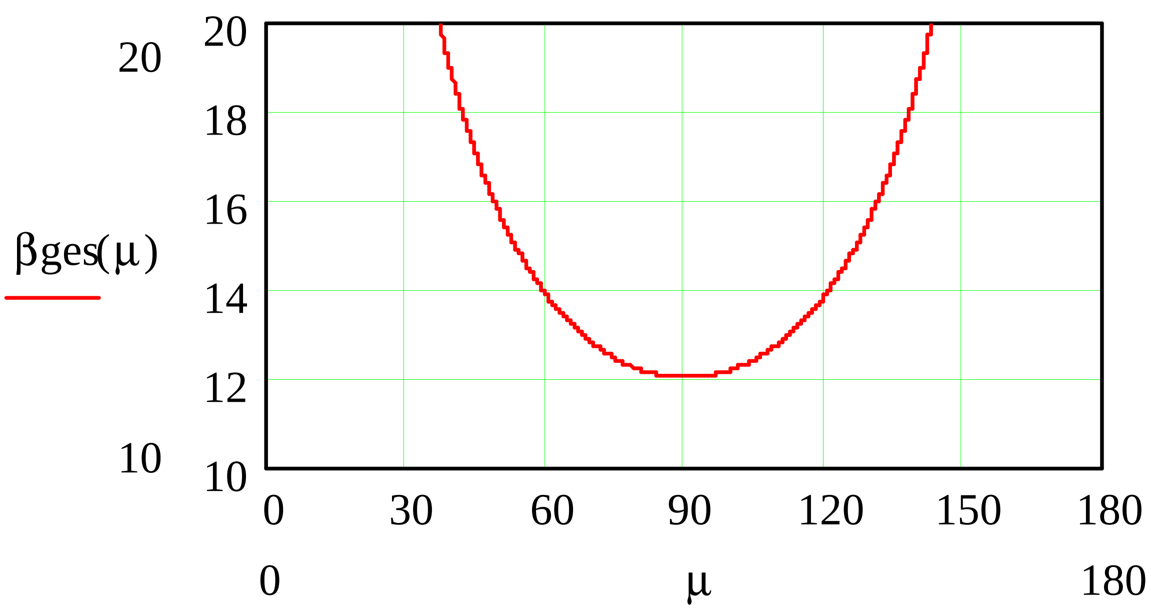
$$r^2 = \varepsilon_x \beta_x + \varepsilon_y \beta_y$$

search for the phase advance μ that results in a minimum of the sum of the beta's

$$\hat{\beta} + \check{\beta} = \frac{(1+\sin\frac{\mu}{2})*L}{\sin\mu} + \frac{(1-\sin\frac{\mu}{2})*L}{\sin\mu}$$

$$\hat{\beta} + \check{\beta} = \frac{2L}{\sin\mu}$$

$$\frac{d}{d\mu}\left(2L/\sin\mu\right) = 0$$



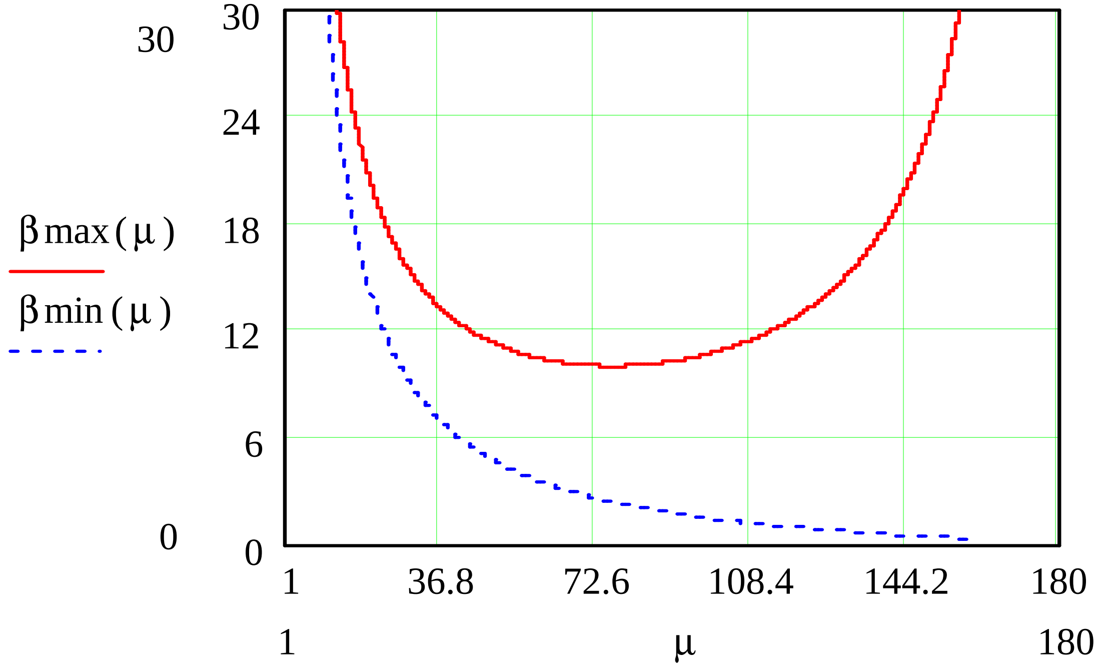
$$\frac{L}{\sin^2\mu} * \cos\mu = 0 \rightarrow \mu = 90°$$

Electrons are different

electron beams are usually flat, $\varepsilon_y \approx 2 - 10\ \%\ \varepsilon_x$

→ ***optimise only*** β_{hor}

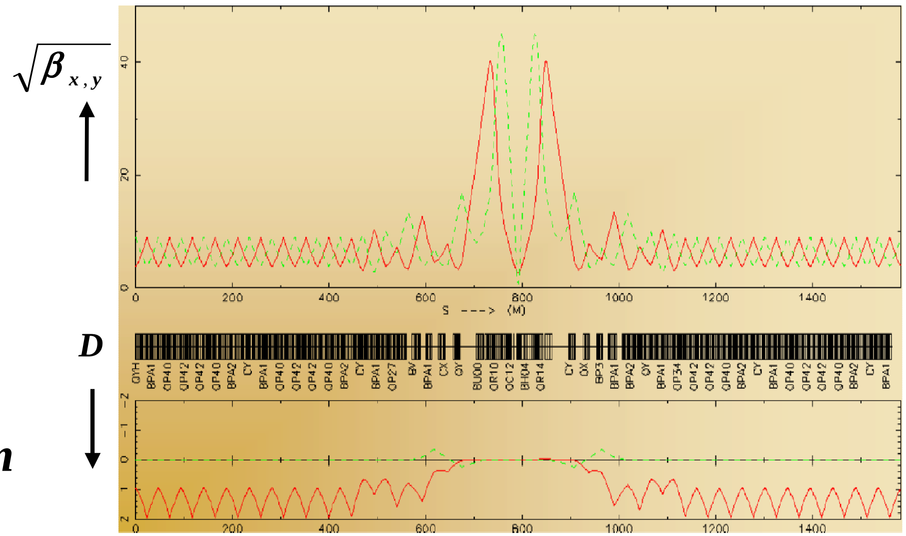
$$\frac{d}{d\mu}(\hat{\beta}) = \frac{d}{d\mu}\frac{L(1+\sin\frac{\mu}{2})}{\sin\mu} = 0 \rightarrow \mu \approx 76°$$



red curve: β_{max}
blue curve: β_{min}
as a function of the phase advance μ

16.) Insertions

The „ not so ideal world “



Question to the audience:

what will happen to the beam parameters α, β, γ if we stop focusing for a while …?

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_S = \begin{pmatrix} C^2 & -2SC & S^2 \\ -CC' & SC'+S'C & -SS' \\ C'^2 & -2S'C' & S'^2 \end{pmatrix} * \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_0$$

… the most complicated insertion: the drift space

$$M = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$$

$$\beta(s) = \beta_0 - 2\alpha_0 s + \gamma_0 s^2$$
$$\alpha(s) = \alpha_0 - \gamma_0 s$$
$$\gamma(s) = \gamma_0$$

„0“ refers to the position of the last lattice element
„s“ refers to the position in the drift

β-Function in a Drift:

let‘s assume we are at a symmetry point in the center of a drift.

$$\beta(s) = \beta_0 - 2\alpha_0 s + \gamma_0 s^2$$

as $\alpha_0 = 0, \quad \rightarrow \quad \gamma_0 = \frac{1+\alpha_0^2}{\beta_0} = \frac{1}{\beta_0}$

and we get for the β function in the neighborhood of the symmetry point

$$\beta(s) = \beta_0 + \frac{s^2}{\beta_0}$$

! ! !

Nota bene:

1.) ***this is very bad !!!***

2.) ***this is a direct consequence of the conservation of phase space density (... in our words: $\varepsilon = const$) … and there is no way out.***

3.) ***Thank you, Mr. Liouville !!!***

Joseph Liouville, 1809-1882

β−Function in a Drift:

If we cannot fight against Liouvuille theorem ... at least we can optimise

Optimisation of the beam dimension:

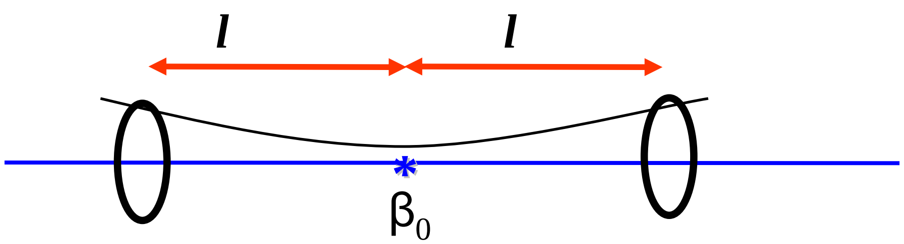
$$\beta(\ell) = \beta_0 + \frac{\ell^2}{\beta_0}$$

Find the β at the center of the drift that leads to the lowest maximum β at the end:

$$\frac{d\hat{\beta}}{d\beta_0} = 1 - \frac{\ell^2}{\beta_0^2} \overset{!}{=} 0$$

$$\rightarrow \beta_0 = \ell$$

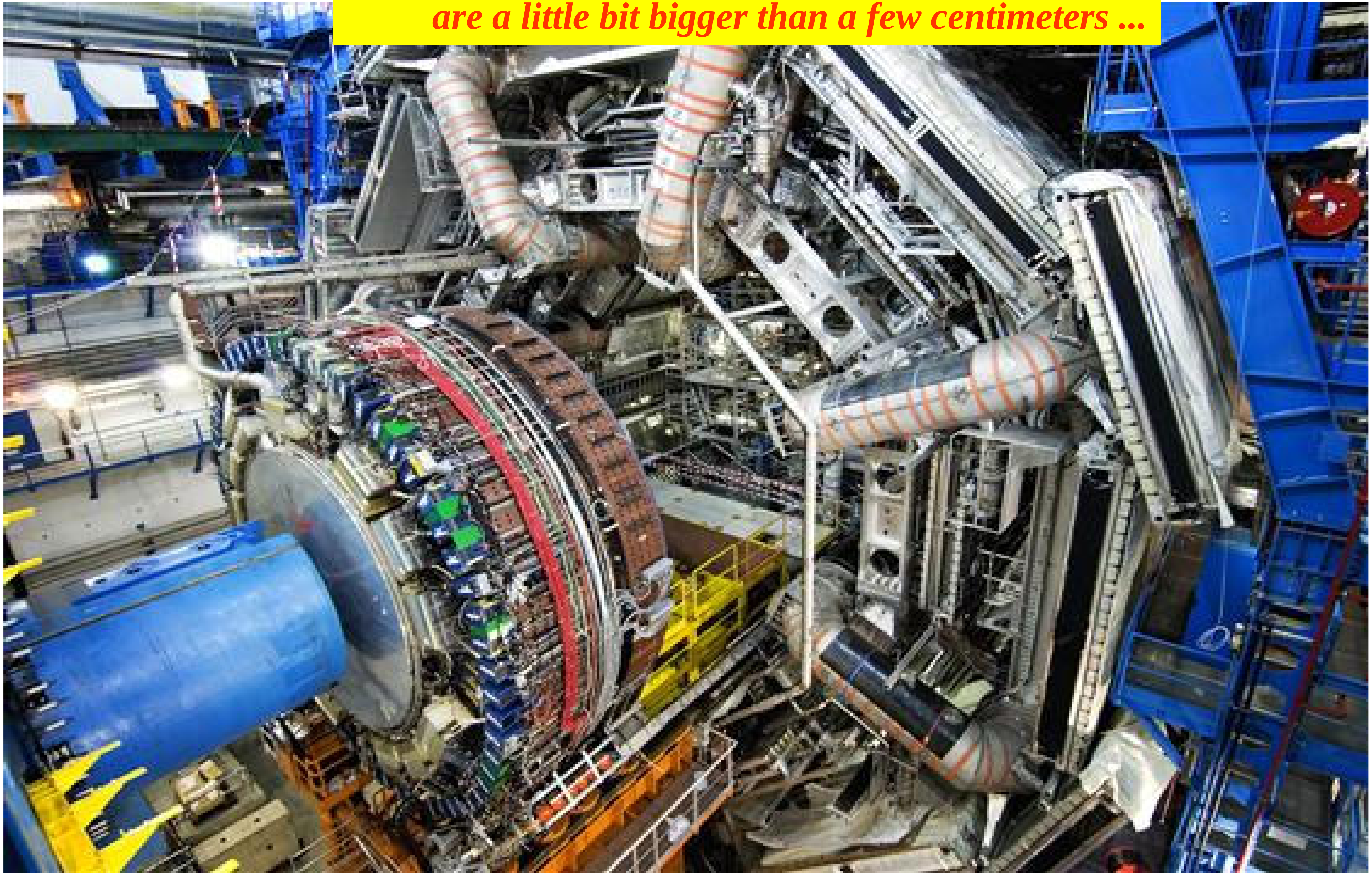
$$\rightarrow \hat{\beta} = 2\beta_0$$



If we choose $\beta_0 = \ell$ we get the smallest β at the end of the drift and the maximum β is just twice the distance ℓ

... clearly there is a

But: ***... unfortunately ... in general***
high energy detectors that are
installed in that drift spaces
are a little bit bigger than a few centimeters ...

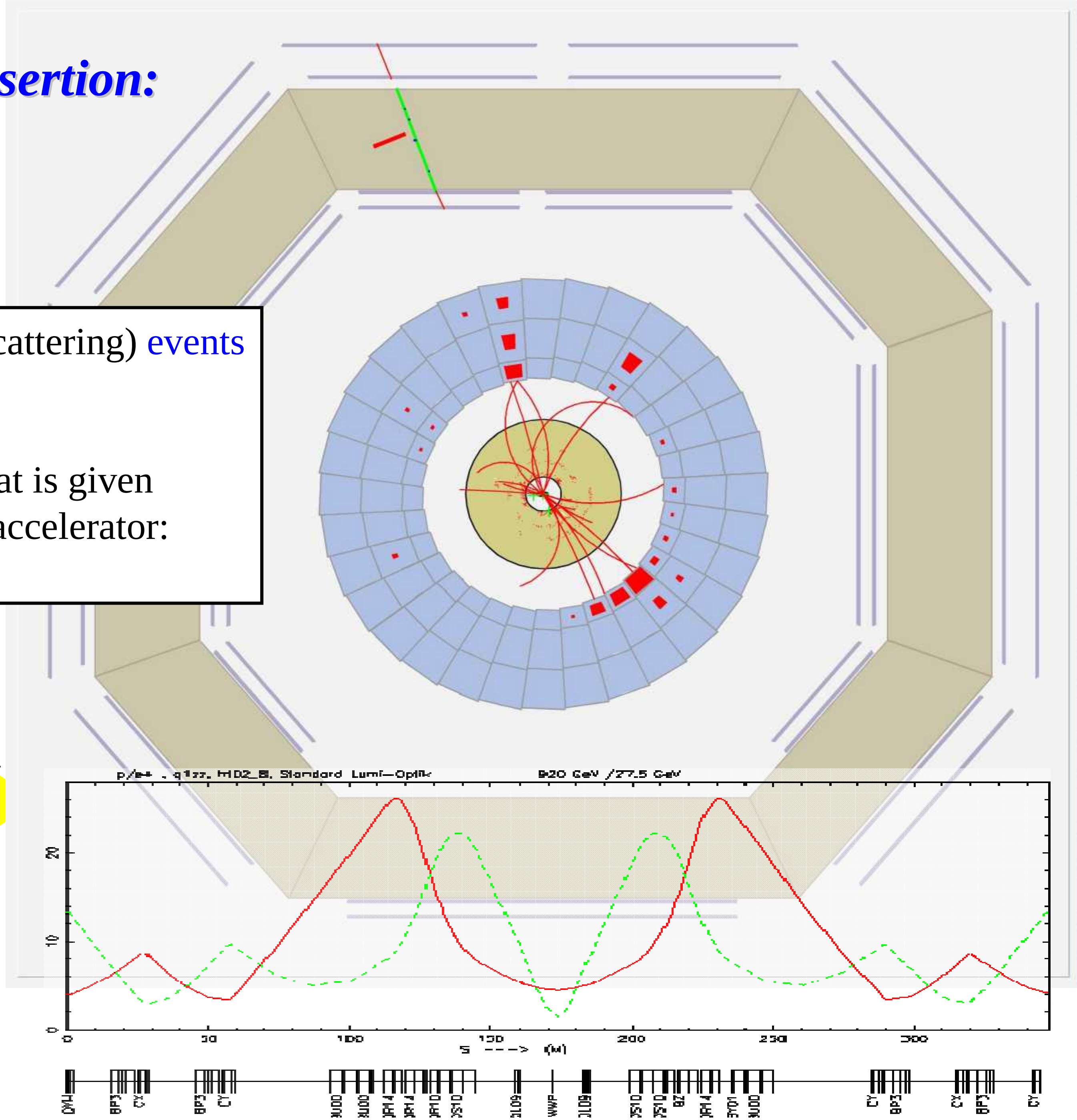


17.) The Mini-β Insertion:

$$R = L * \Sigma_{react}$$

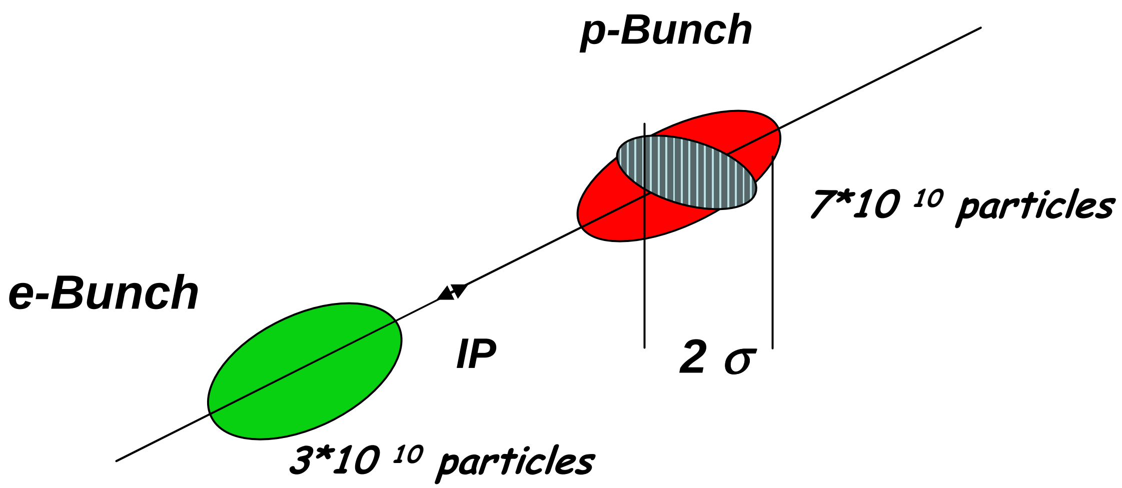
production rate of (scattering) events is determined by the cross section Σ_{react} and a parameter L that is given by the design of the accelerator:
… the luminosity

$$L = \frac{1}{4\pi e^2 f_0 \, b} * \frac{I_1 * I_2}{\sigma^*_x * \sigma^*_y}$$



ZEUS detector: inelastic scattering event of e+/p

Luminosity



Example: e- p collider run at HERA

$\beta_x = 2.45\, m$ $\beta_y = 0.18\, m$

$\varepsilon_x = 7 * 10^{-9}\ rad\ m$ $\varepsilon_x = \varepsilon_y$

$\sigma_x = 118\ \mu m$ $\sigma_x = 32\ \mu m$

$I_e = 43\, mA$ $f_0 = 47.3\, kHz$

$I_p = 84\, mA$ $n_b = 180$

$$L = \frac{1}{4\pi e^2 f_0 n_b} * \frac{I_e I_p}{\sigma_x \sigma_y}$$

$$L = 34.0 * 10^{30}\ {}^{1}\!/_{cm^2 s}$$

Mini-β Insertions: Betafunctions

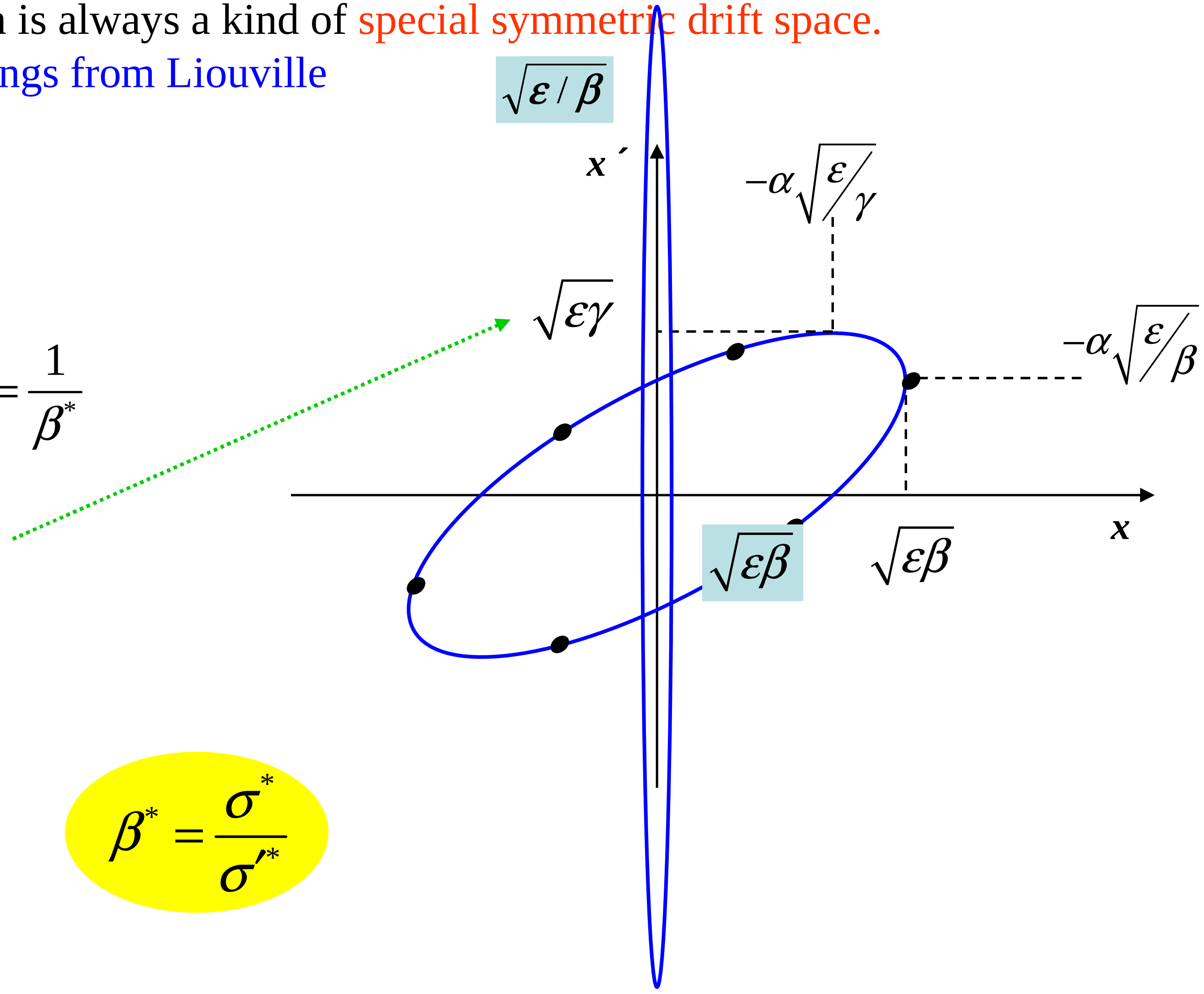
A mini-β insertion is always a kind of special symmetric drift space.
→greetings from Liouville

$$\alpha^* = 0$$

$$\gamma^* = \frac{1+\alpha^2}{\beta} = \frac{1}{\beta^*}$$

$$\sigma'^* = \sqrt{\frac{\varepsilon}{\beta^*}}$$

$$\beta^* = \frac{\sigma^*}{\sigma'^*}$$



at a symmetry point β is just the ratio of beam dimension and beam divergence.

Mini Beta Insertions: some guide lines

* Calculate the periodic solution in the arc

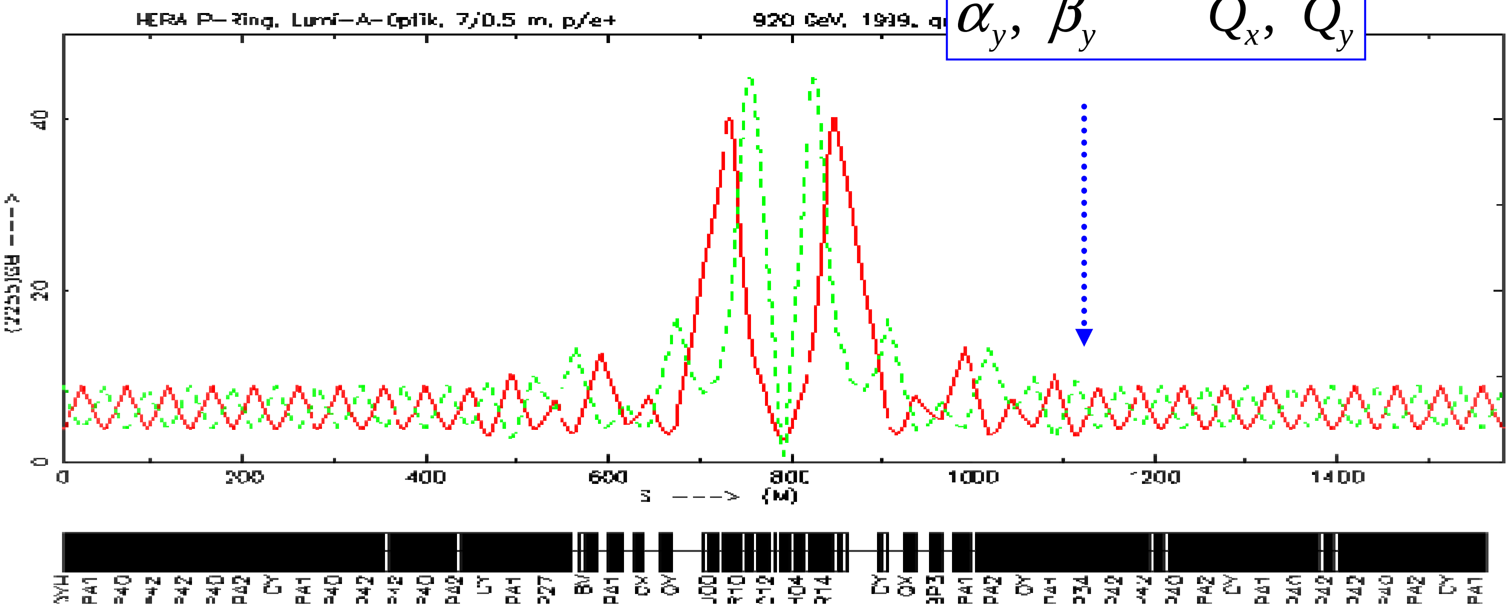
* Introduce the drift space needed for the insertion device (e.g. particle detector)

* Install a quadrupole dublet (or triplet) as close as possible

* Introduce additional quadrupol magnets to match the beam parameters To the values at the beginning of the next arc structure.

$$\alpha_x,\ \beta_x \qquad D_x,\ D_y$$
$$\alpha_y,\ \beta_y \qquad Q_x,\ Q_y$$

8 individually powered quad magnets are needed to match the insertion
(... at least)



Are there any problems ??

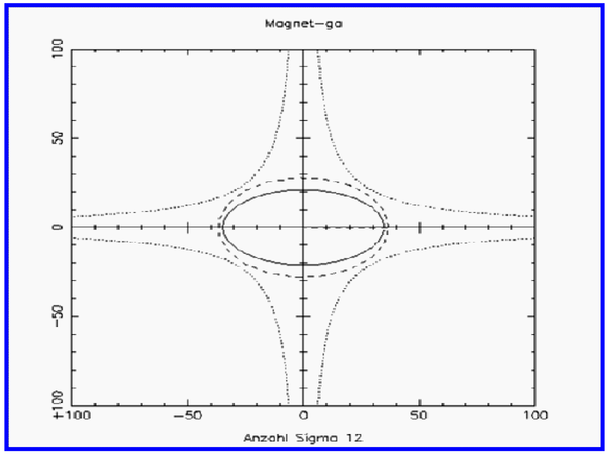
sure there are...

**** large β values at the doublet quadrupoles → large contribution to chromaticity Q' … and no local correction***

$$\xi = \frac{1}{4\pi} \oint \{K(s)\beta(s)\} ds$$

**** aperture of mini β quadrupoles limit the luminosity***

beam envelope at the first
mini β quadrupole lens in
the HERA proton storage ring



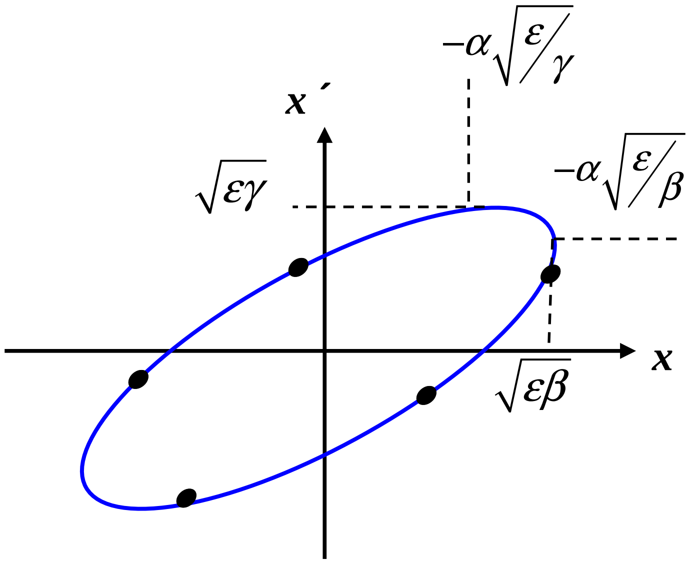
**** field quality and magnet stability most critical at the high β sections***
effect of a quad error:

$$\Delta Q = \int_{s0}^{s0+l} \frac{\Delta K(s)\beta(s) ds}{4\pi}$$

→ keep distance „s“ to the first mini β quadrupole as small as possible

18.) Liouville during Acceleration

$$\varepsilon = \gamma(s)\, x^2(s) + 2\alpha(s) x(s) x'(s) + \beta(s)\, x'^2(s)$$



Beam Emittance ***corresponds to the area covered in the x, x´Phase Space Ellipse***

Liouville: ***Area in phase space is constant.***

But so sorry ... $\varepsilon \neq$ const !

Classical Mechanics:

phase space ***= diagram of the two canonical variables***
position   & momentum
x p_x

$$p_j = \frac{\partial L}{\partial \dot{q}_j} \quad ; \quad L = T - V = kin.\,Energy - pot.\,Energy$$

According to Hamiltonian mechanics:
phase space diagram relates the variables q and p

q = position = x
p = momentum = mc $\gamma\beta_x$

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \quad ; \quad \beta_x = \frac{\dot{x}}{c}$$

Liouvilles Theorem: $\int p\,dq = const$

for convenience (i.e. because we are lazy bones) we use in accelerator theory:

$$x' = \frac{dx}{ds} = \frac{dx}{dt}\frac{dt}{ds} = \frac{\beta_x}{\beta}$$

where $\beta_x = v_x / c$

$$\int pdq = mc\int \gamma\beta_x dx$$

$$\int pdq = mc\gamma\beta \underbrace{\int x'dx}_{\varepsilon}$$

$$\Rightarrow \quad \varepsilon = \int x'dx \propto \frac{1}{\beta\gamma}$$

the beam emittance shrinks during acceleration $\varepsilon \sim 1/\gamma$

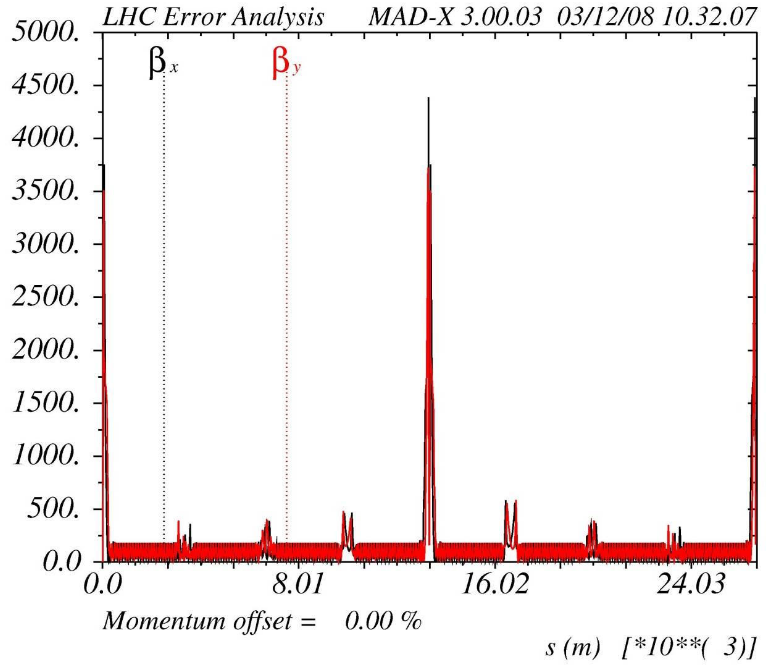
Nota bene:

1.) *A proton machine … or an electron linac … needs the highest aperture at injection energy !!!*
as soon as we start to accelerate the beam size shrinks as $\gamma^{-1/2}$ *in both planes.*

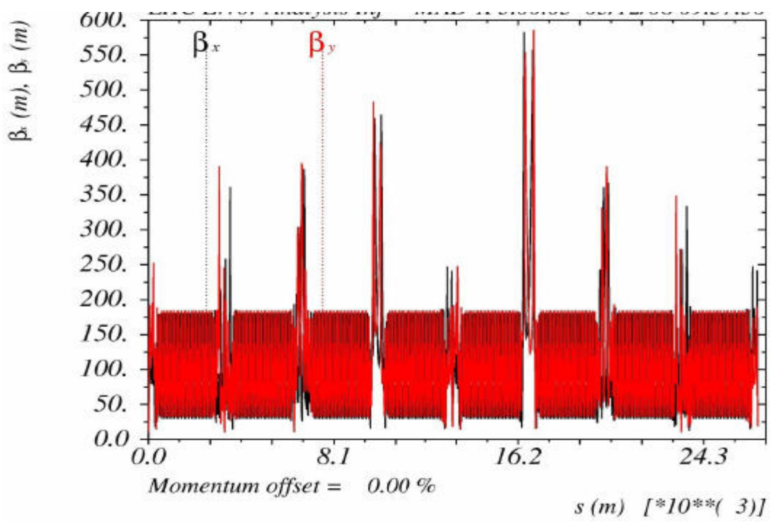
$$\sigma = \sqrt{\varepsilon\beta}$$

2.) *At lowest energy the machine will have the major aperture problems,*
→ *here we have to minimise* $\hat{\beta}$

3.) *we need different beam optics adopted to the energy:*
A Mini Beta concept will only be adequate at flat top.



LHC mini beta optics at 7000 GeV

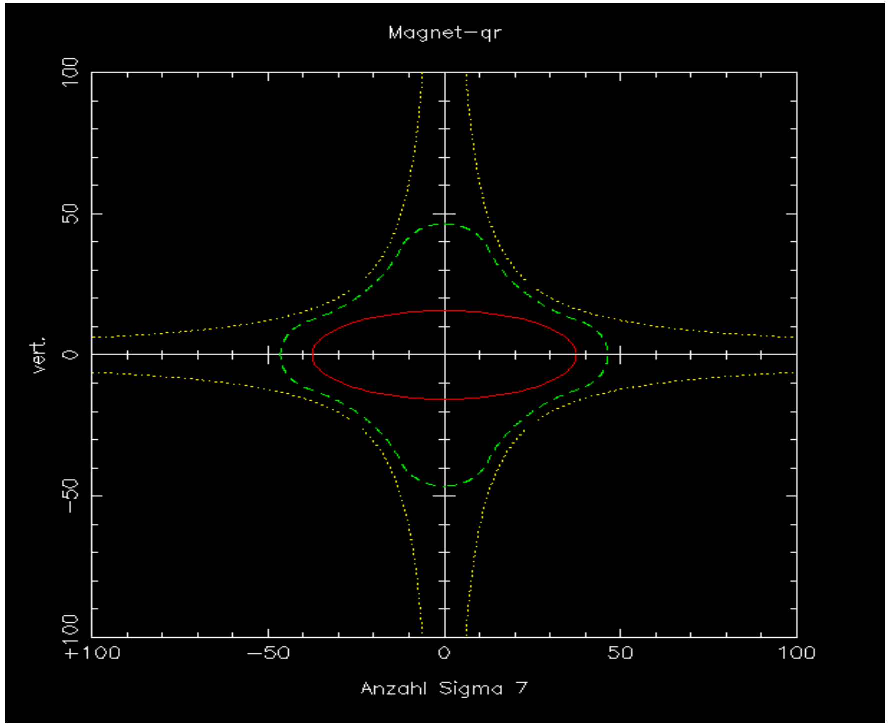


LHC injection optics at 450 GeV

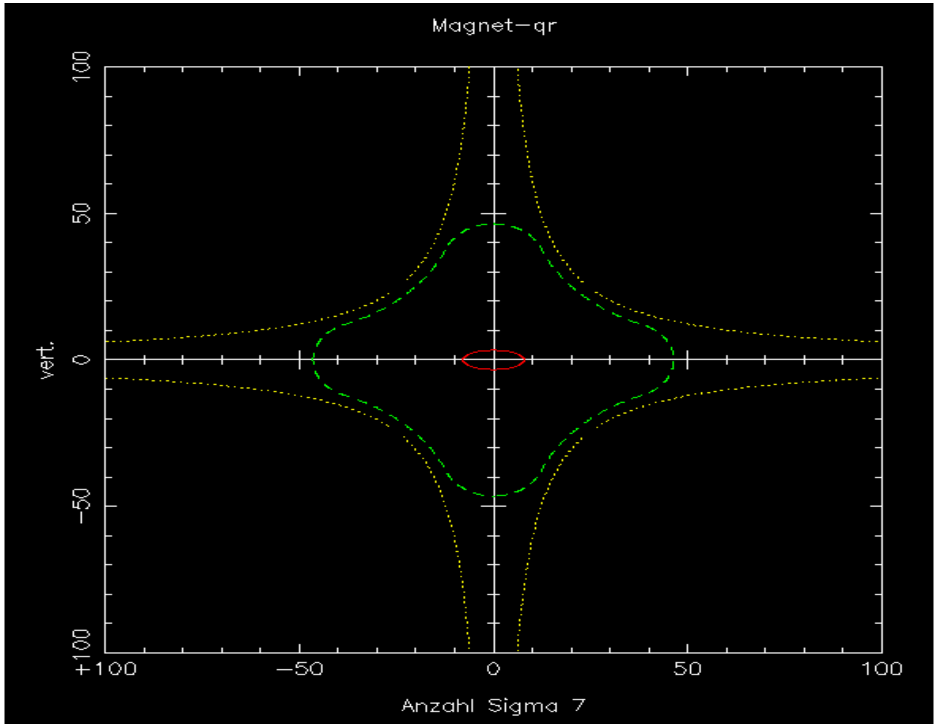
Example: HERA proton ring

injection energy: 40 GeV $\gamma = 43$
flat top energy: 920 GeV $\gamma = 980$

emittance $\varepsilon\,(40GeV) = 1.2 * 10^{-7}$
$\varepsilon\,(920GeV) = 5.1 * 10^{-9}$

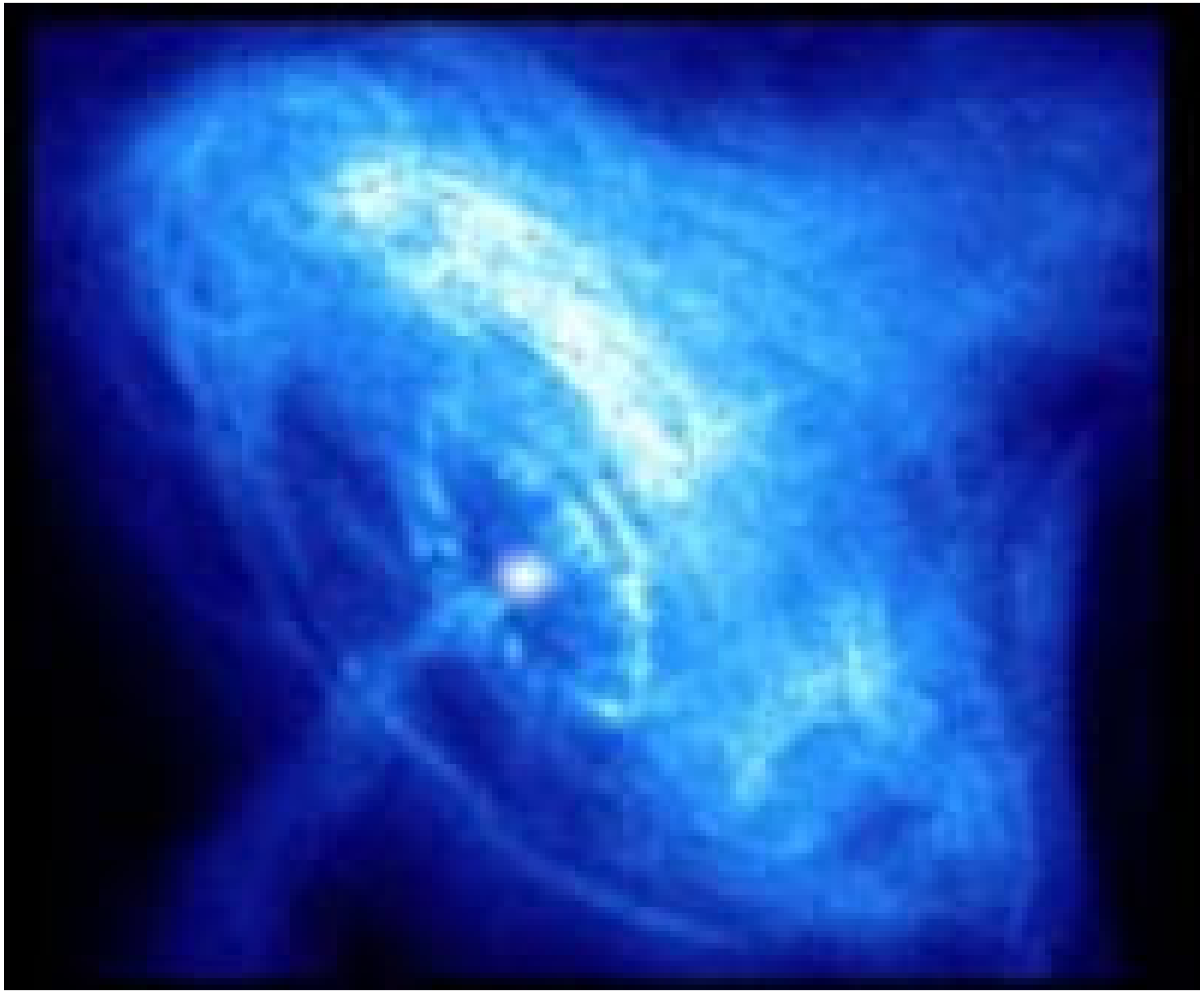


7 σ beam envelope at E = 40 GeV



… and at E = 920 GeV

… let´s talk about acceleration



crab nebula,

burst of charged particles E = 10 ^{20} eV

The „ not so ideal world “

19.) The „ $\Delta p / p \neq 0$“ Problem

ideal accelerator: ***all particles will see the same accelerating voltage.***

$\rightarrow \Delta p / p = 0$

„nearly ideal“ accelerator: *Cockroft Walton or van de Graaf*

$\Delta p / p \approx 10^{-5}$

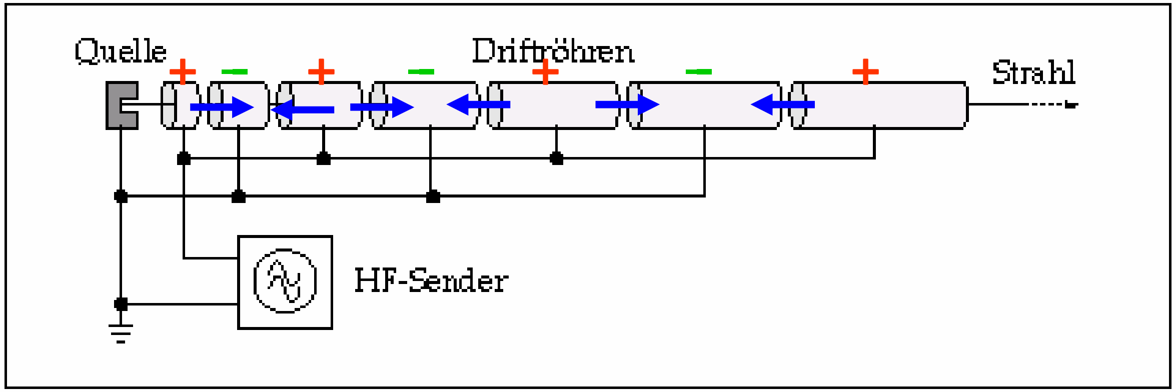
Vivitron, Straßbourg, inner structure of the acc. section

MP Tandem van de Graaf Accelerator at MPI for Nucl. Phys. Heidelberg

Linear Accelerator

1928, Wideroe

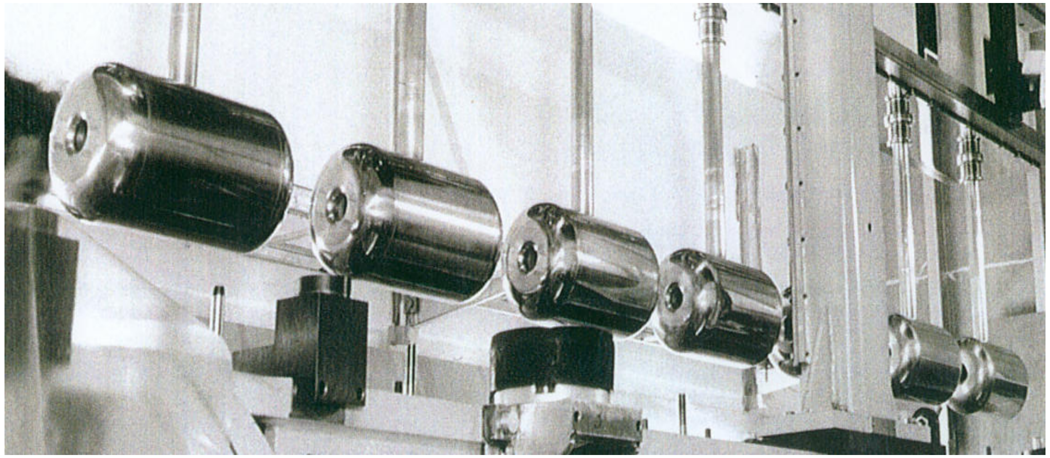
schematic Layout:



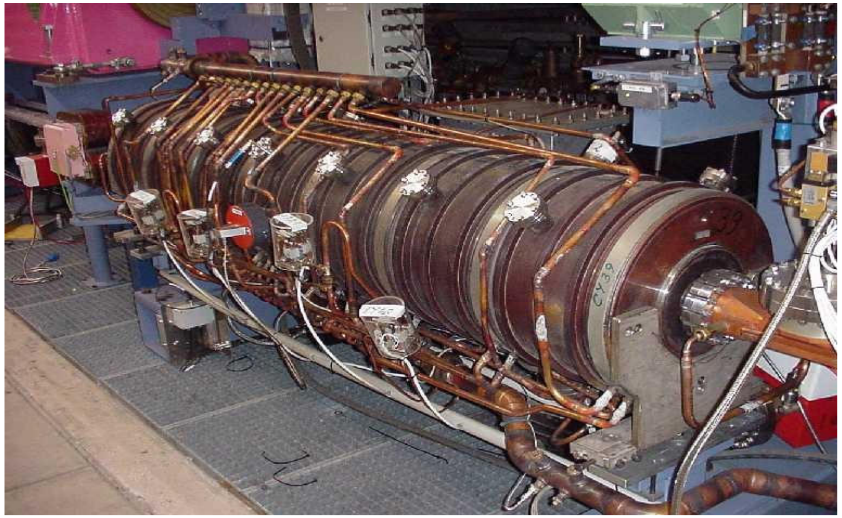
Energy Gain per „Gap":

$$W = q * U_0 * \sin \omega_{RF} t$$

drift tube structure at a proton linac



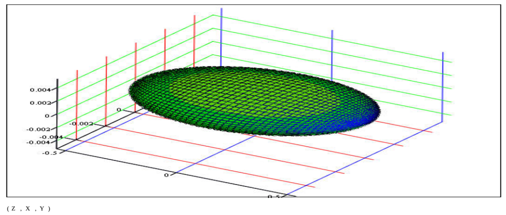
500 MHz cavities in an electron storage ring



* *RF Acceleration:* multiple application of the same acceleration voltage; brillant idea to gain higher energies

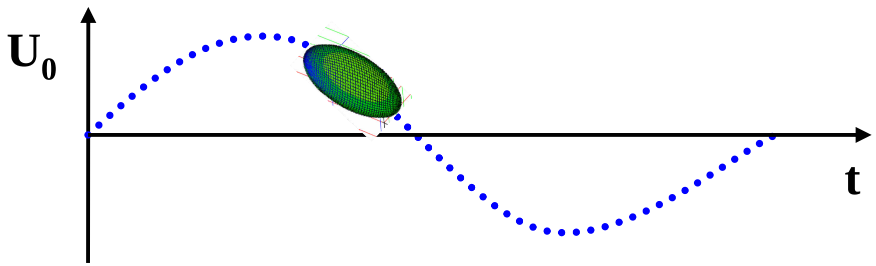
Problem: panta rhei !!!

(Heraklit: 540-480 v. Chr.)



Bunch length of Electrons ≈ 1cm

Example: HERA RF:



$$\lambda = 60\ cm$$

$$\nu = 500\ MHz$$
$$c = \lambda * \nu$$
$$\lambda = 60\ cm$$

$$\sin(90^o) = 1$$
$$\sin(84^o) = 0.994$$

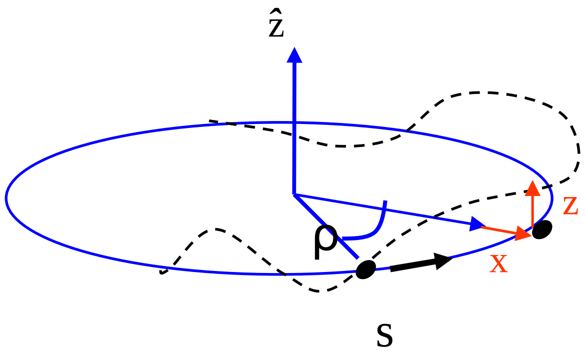
$$\frac{\Delta U}{U} = 6*10^{-3}$$

typical momentum spread of an electron bunch:

$$\frac{\Delta p}{p} \approx 1*10^{-3}$$

20.) Dispersion: trajectories for $\Delta p / p \neq 0$

Question: *do you remember last session, page 11 ? … sure you do*



Force acting on the particle

$$F = m\frac{d^2}{dt^2}(x+\rho) - \frac{mv^2}{x+\rho} = -e\,B_z\,v$$

remember: x ≈ mm , ρ ≈ m … → develop for small x

$$m\frac{d^2x}{dt^2} - \frac{mv^2}{\rho}(1-\frac{x}{\rho}) = -eB_z v$$

consider only linear fields, and change independent variable: t → s

$$B_z = B_0 + x\frac{\partial B_z}{\partial x}$$

$$x'' - \frac{1}{\rho}(1-\frac{x}{\rho}) = \frac{-e\ B_0}{mv} - \frac{e\ x\ g}{mv}$$

$p = p_0 + \Delta p$

… but now take a small momentum error into account !!!

Dispersion:

develop for small momentum error

$$\Delta p \ll p_0 \Rightarrow \frac{1}{p_0 + \Delta p} \approx \frac{1}{p_0} - \frac{\Delta p}{p_0^2}$$

$$x'' - \frac{1}{\rho} + \frac{x}{\rho^2} \approx \underbrace{\frac{-e\ B_0}{p_0}}_{-\frac{1}{\rho}} + \frac{\Delta p}{p_0^2} eB_0 - \underbrace{\frac{xeg}{p_0}}_{k * x} + \underbrace{xeg\ \frac{\Delta p}{p_0^2}}_{\approx 0}$$

$$x'' + \frac{x}{\rho^2} - kx = \frac{\Delta p}{p_0} * \frac{1}{\rho}$$

$$x'' + x\left(\frac{1}{\rho^2} - k\right) = \frac{\Delta p}{p_0} * \frac{1}{\rho}$$

Momentum spread of the beam adds a term on the r.h.s. of the equation of motion.
→ inhomogeneous differential equation.

Dispersion:

$$x'' + x\left(\frac{1}{\rho^2} - k\right) = \frac{\Delta p}{p} \cdot \frac{1}{\rho}$$

general solution:

$$x(s) = x_h(s) + x_i(s)$$

$$\begin{cases} x_h''(s) + K(s) \cdot x_h(s) = 0 \\ x_i''(s) + K(s) \cdot x_i(s) = \frac{1}{\rho} \cdot \frac{\Delta p}{p} \end{cases}$$

Normalise with respect to Δp/p:

$$D(s) = \frac{x_i(s)}{\Delta p / p}$$

Dispersion function D(s)

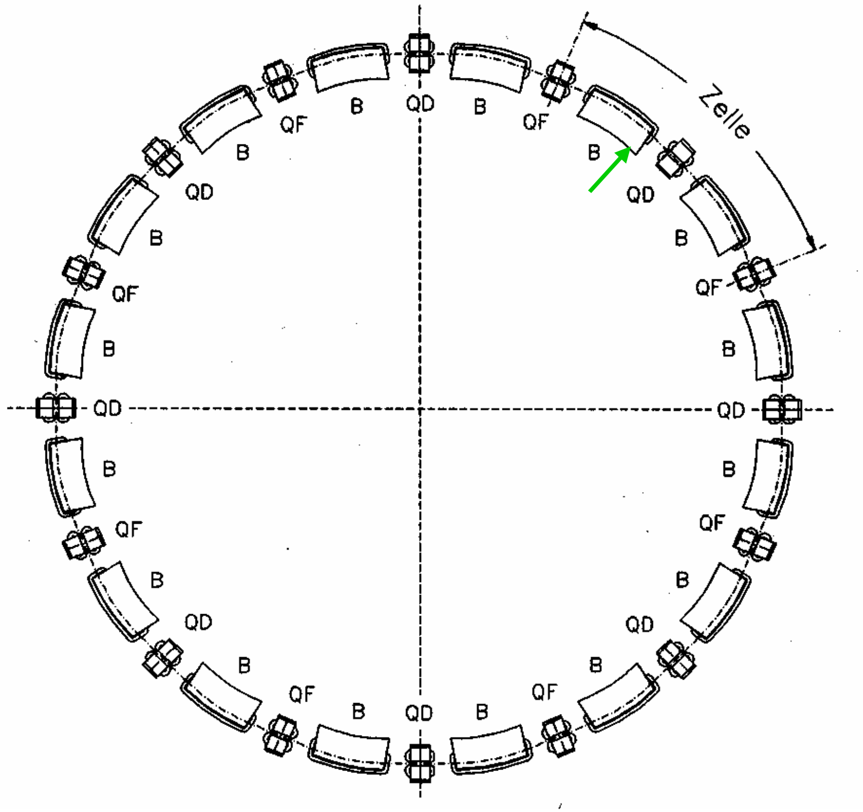
**** is that special orbit, an ideal particle would have for Δp/p = 1***

**** the orbit of any particle is the sum of the well known x_β and the dispersion***

**** as D(s) is just another orbit it will be subject to the focusing properties of the lattice***

Dispersion

Example: homogenous dipole field



bit for $\Delta p/p > 0$

$$= D(s) \cdot \frac{\Delta p}{p}$$

Matrix formalism:

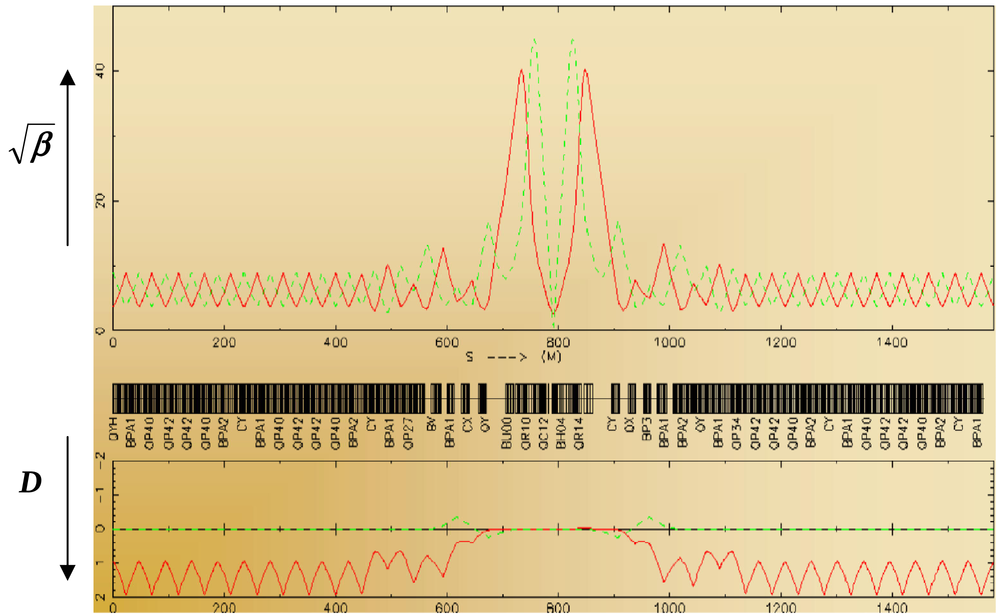
$$x(s) = x_\beta(s) + D(s) \cdot \frac{\Delta p}{p}$$

$$x(s) = C(s) \cdot x_0 + S(s) \cdot x'_0 + D(s) \cdot \frac{\Delta p}{p}$$

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_0 + \frac{\Delta p}{p} \begin{pmatrix} D \\ D' \end{pmatrix}$$

or expressed as 3x3 matrix

$$\begin{pmatrix} x \\ x' \\ \Delta p/p \end{pmatrix}_s = \begin{pmatrix} C & S & D \\ C' & S' & D' \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ x' \\ \Delta p/p \end{pmatrix}_0$$



Example HERA

$$\left. \begin{aligned} x_\beta &= 1 \dots 2\ mm \\ D(s) &\approx 1 \dots 2\ m \\ \Delta p/p &\approx 1 \cdot 10^{-3} \end{aligned} \right\}$$

Amplitude of Orbit oscillation

contribution due to Dispersion ≈ beam size

→ Dispersion must vanish at the collision point

!

Calculate D, D´

$$D(s) = S(s)\int_{s0}^{s1} \frac{1}{\rho} C(\tilde{s})d\tilde{s} - C(s)\int_{s0}^{s1} \frac{1}{\rho} S(\tilde{s})d\tilde{s}$$

(proof: see appendix)

Exampel: Drift

$$D(s) = S(s)\underbrace{\int_{s0}^{s1}\frac{1}{\rho}C(\tilde{s})d\tilde{s}}_{=0} - C(s)\underbrace{\int_{s0}^{s1}\frac{1}{\rho}S(\tilde{s})d\tilde{s}}_{=0}$$

$$M_{drift} = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix} \qquad M_{Drift} = \begin{pmatrix} 1 & l & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Example: Dipole

Remember: ***Matrix of a magnetic element***

$$M_{foc} = \begin{pmatrix} \cos(\sqrt{|K|}l) & \frac{1}{\sqrt{|K|}}\sin(\sqrt{|K|}l) \\ -\sqrt{|K|}\sin(\sqrt{|K|}l) & \cos(\sqrt{|K|}l) \end{pmatrix}$$

$K = k - \frac{1}{\rho^2}$ ***… but in a dipole, as $k = 0$ …***

$$M_{foc} = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} = \begin{pmatrix} \cos\frac{l}{\rho} & \rho\sin\frac{l}{\rho} \\ -\frac{1}{\rho}\sin\frac{l}{\rho} & \cos\frac{l}{\rho} \end{pmatrix}$$

Example: *Dispersion in a Sector Dipole Magnet*

calculate the „D“ elements of the marix

$$D(s) = S(s)\int_{s0}^{s1}\frac{1}{\rho}C(\tilde{s})d\tilde{s} - C(s)\int_{s0}^{s1}\frac{1}{\rho}S(\tilde{s})d\tilde{s}$$

$$D(s) = (\rho\sin\frac{l}{\rho})*\frac{1}{\rho}*(\rho\sin\frac{l}{\rho}) - \cos\frac{l}{\rho}*\frac{1}{\rho}*\rho\cdot(-\cos\frac{l}{\rho}+1)*\rho$$

$$D(s) = \rho\sin^2\frac{l}{\rho} + \rho\cos\frac{l}{\rho}*(\cos\frac{l}{\rho}-1)$$

$$D(s) = \rho\cdot(1-\cos\frac{l}{\rho}) \qquad D'(s) = \sin\frac{l}{\rho}$$

Dispersion elements in a sector dipole magnet

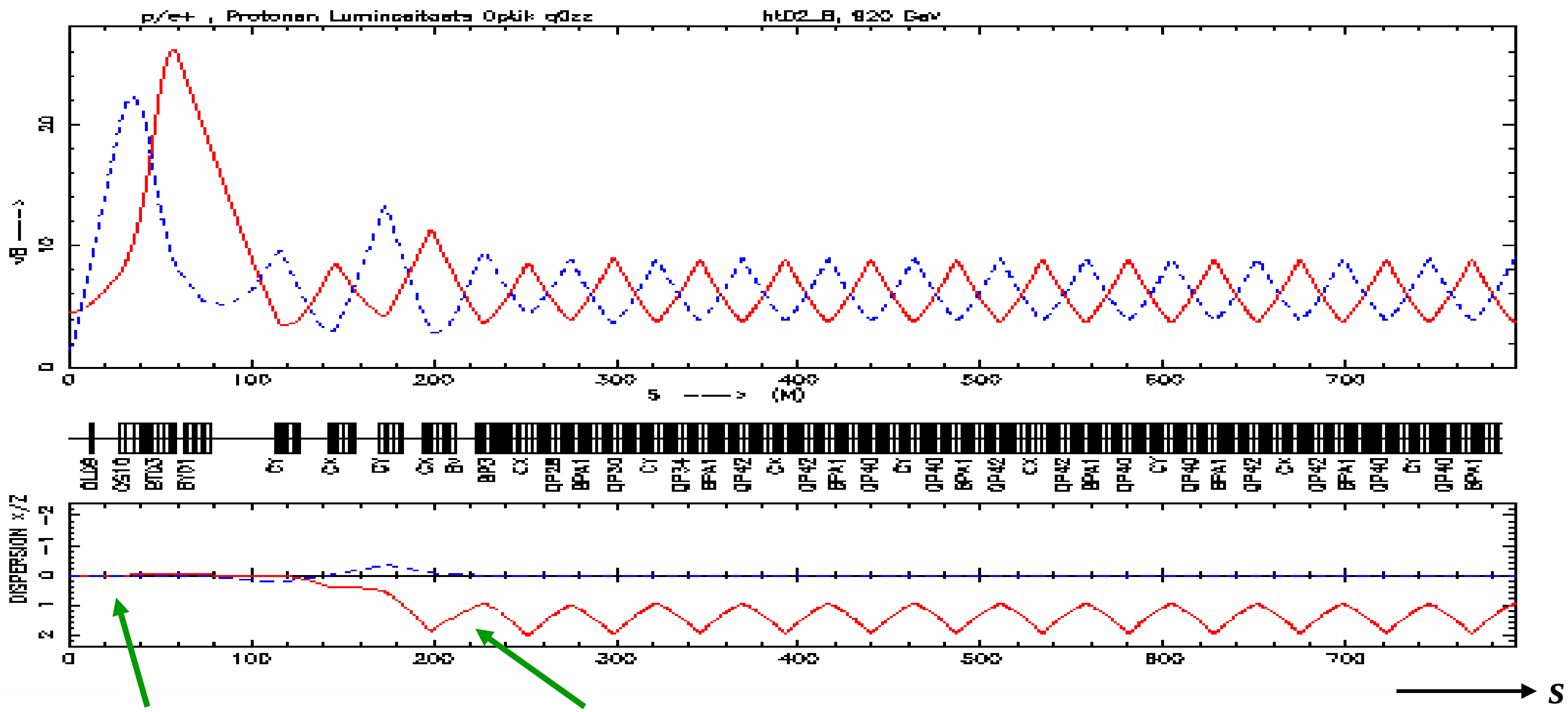
$$\begin{pmatrix} x \\ x' \\ \Delta p/p \end{pmatrix}_{s2} = \begin{pmatrix} \cos\frac{l}{\rho} & \rho\sin\frac{l}{\rho} & \rho*(1-\cos\frac{l}{\rho}) \\ -\frac{1}{\rho}\sin\frac{l}{\rho} & \cos\frac{l}{\rho} & \sin\frac{l}{\rho} \\ 0 & 0 & 1 \end{pmatrix} * \begin{pmatrix} x \\ x' \\ \Delta p/p \end{pmatrix}_{s1}$$

Example: Dispersion, calculated by an optics code for a real machine

$$x_D = D(s) * \frac{\Delta p}{p}$$

* *D(s) is created by the dipole magnets*

… and afterwards focused by the quadrupole fields

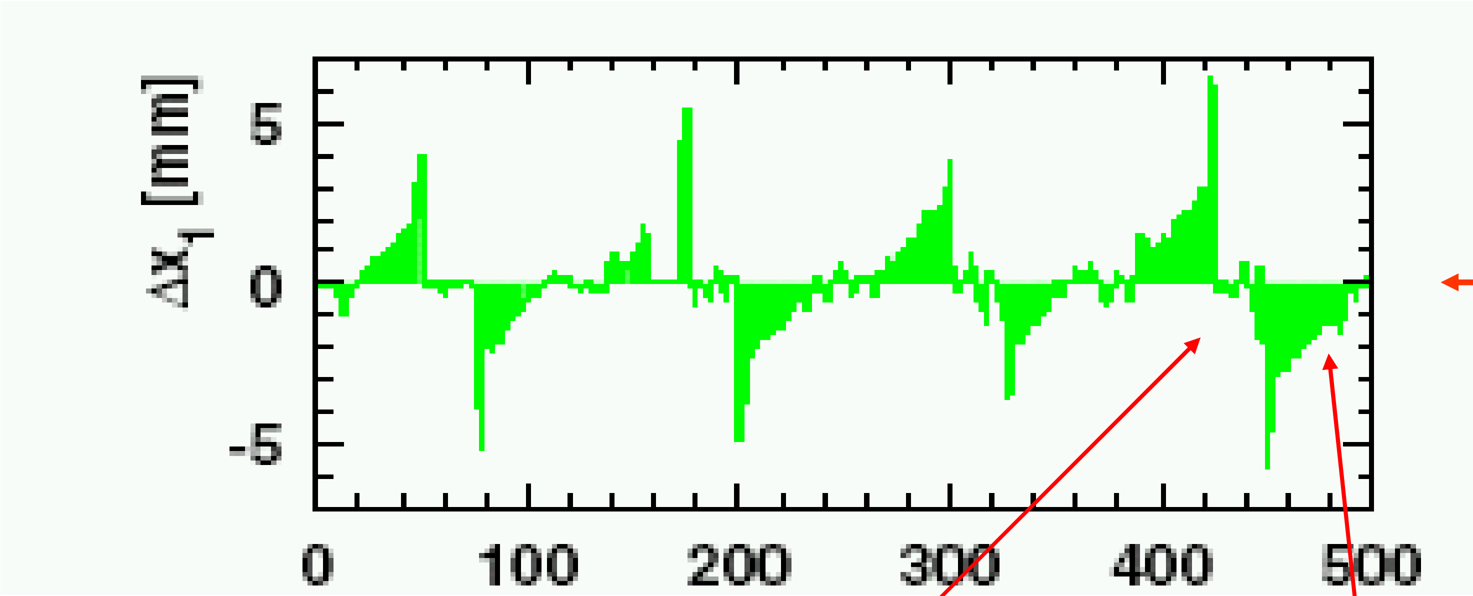


Mini Beta Section,
→ no dipoles !!!

D(s) ≈ 1 … 2 m

Periodic Dispersion:

„Sawtooth Effect“ at LEP (CERN)



Electron course

BPM Number

In the straight sections they are accelerated by the rf cavities so much that they „overshoot“ and reach nearly the outer side of the vacuum chamber.

In the arc the electron beam loses so much energy in each octant that the particle are running more and more on a dispersion trajectory.

21.) Momentum Compaction Factor:

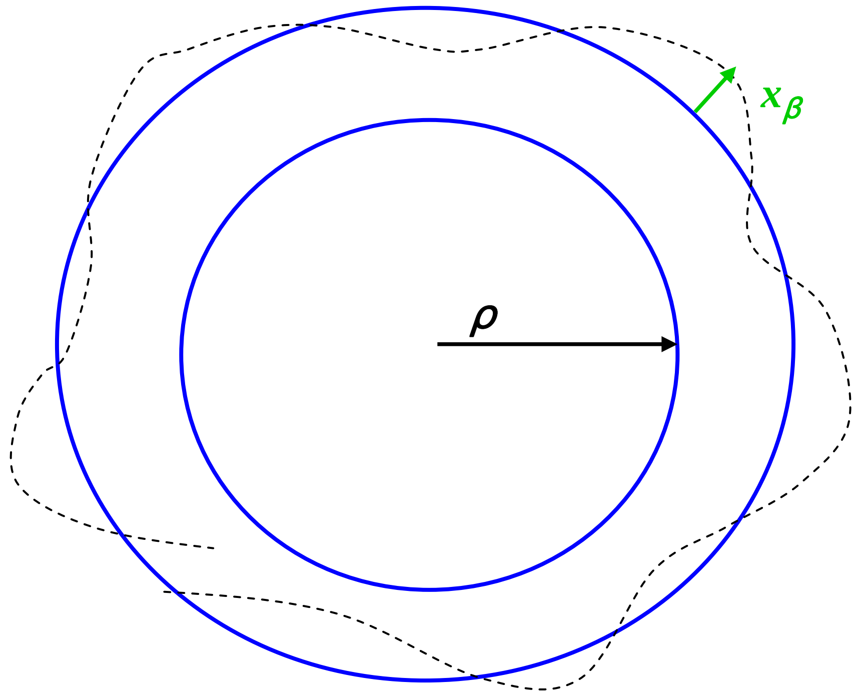
The dispersion function relates the momentum error of a particle to the horizontal orbit coordinate.

inhomogeneous differential equation

$$x'' + K(s) * x = \frac{1}{\rho} \frac{\Delta p}{p}$$

general solution

$$x(s) = x_\beta(s) + D(s) \frac{\Delta p}{p}$$



But it does much more:

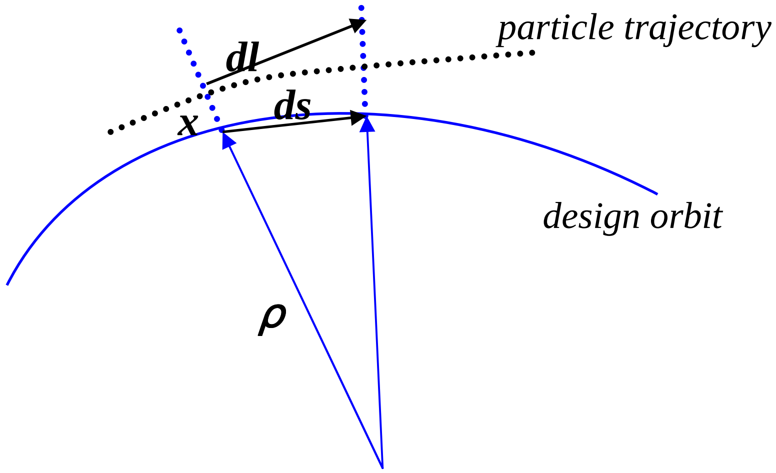
it changes the length of the off - energy - orbit !!

Momentum Compaction Factor: α_p

particle with a displacement x to the design orbit
→ path length dl ...

$$\frac{dl}{ds} = \frac{\rho + x}{\rho}$$

$$\rightarrow \quad dl = \left(1 + \frac{x}{\rho(s)}\right) ds$$



circumference of an off-energy closed orbit

$$l_{\Delta E} = \oint dl = \oint \left(1 + \frac{x_{\Delta E}}{\rho(s)}\right) ds$$

remember:

$$x_{\Delta E}(s) = D(s)\frac{\Delta p}{p}$$

$$\delta l_{\Delta E} = \frac{\Delta p}{p} \oint \left(\frac{D(s)}{\rho(s)}\right) ds$$

**** The lengthening of the orbit for off-momentum particles is given by the dispersion function and the bending radius.***

Definition:

$$\frac{\delta l_\varepsilon}{L} = \alpha_{cp} \frac{\Delta p}{p}$$

$$\rightarrow \quad \alpha_{cp} = \frac{1}{L} \oint \left(\frac{D(s)}{\rho(s)} \right) ds$$

For first estimates assume:

$$\frac{1}{\rho} = const$$

$$\int_{dipoles} \boldsymbol{D(s)ds} = \Sigma\, (\boldsymbol{l}_{dipoles}) * \langle \boldsymbol{D} \rangle_{dipole}$$

$$\alpha_{cp} = \frac{1}{L} l_{dipoles} \langle D \rangle \frac{1}{\rho} = \frac{1}{L} 2\pi\rho \langle D \rangle \frac{1}{\rho} \quad \rightarrow \quad \alpha_{cp} \approx \frac{2\pi}{L} \langle D \rangle \approx \frac{\langle D \rangle}{R}$$

Assume: $v \approx c$

$$\rightarrow \frac{\delta T}{T} = \frac{\delta l_\varepsilon}{L} = \alpha_{cp} \frac{\Delta p}{p}$$

α_{cp} combines via the dispersion function the momentum spread with the longitudinal motion of the particle.

Resume :

beam emittance

$$\varepsilon \propto \frac{1}{\beta\gamma}$$

beta function in a drift

$$\beta(s) = \beta_0 - 2\alpha_0 s + \gamma_0 s^2$$

… and for $\alpha = 0$

$$\beta(s) = \beta_0 + \frac{s^2}{\beta_0}$$

particle trajectory for $\Delta p/p \neq 0$
inhomogenious equation

$$x'' + x\left(\frac{1}{\rho^2} - k\right) = \frac{\Delta p}{p_0} * \frac{1}{\rho}$$

… and ist solution

$$x(s) = x_\beta(s) + D(s) \cdot \frac{\Delta p}{p}$$

momentum compaction

$$\frac{\delta l_\varepsilon}{L} = \alpha_{cp} \frac{\Delta p}{p}$$

$$\alpha_{cp} \approx \frac{2\pi}{L}\langle D \rangle \approx \frac{\langle D \rangle}{R}$$

Appendix:

Solution of the inhomogenious equation of motion

Ansatz:

$$D(s) = S(s)\int_{s0}^{s1}\frac{1}{\rho}C(\tilde{s})d\tilde{s} - C(s)\int_{s0}^{s1}\frac{1}{\rho}S(\tilde{s})d\tilde{s}$$

$$D'(s) = S'*\int\frac{1}{\rho}C\,dt + S\frac{1}{\rho}C - C'*\int\frac{1}{\rho}S\,dt - C\frac{1}{\rho}S$$

$$D'(s) = S'*\int\frac{C}{\rho}dt \;-\; C'*\int\frac{S}{\rho}dt$$

$$D''(s) = S''*\int\frac{C}{\rho}d\tilde{s} + S'\frac{C}{\rho} \;-\; C''*\int\frac{S}{\rho}d\tilde{s} \;-\; C'\frac{S}{\rho}$$

$$= S''*\int\frac{C}{\rho}d\tilde{s} - C''*\int\frac{S}{\rho}d\tilde{s} + \frac{1}{\rho}\underbrace{(CS' - S\,C')}_{= \det M = 1}$$

---

remember: for Cs) and S(s) to be independent solutions the Wronski determinant has to meet the condition

$$W = \begin{vmatrix} C & S \\ C' & S' \end{vmatrix} \neq 0$$

and as it is independent of the variable „s"

$$\frac{dW}{ds} = \frac{d}{ds}(CS' - SC') = CS'' - SC'' = -K(CS - SC) = 0$$

we get for the initital conditions that we had chosen …

$$\left.\begin{array}{ll} C_0 = 1, & C_0' = 0 \\ S_0 = 0, & S_0' = 1 \end{array}\right\} \qquad W = \begin{vmatrix} C & S \\ C' & S' \end{vmatrix} = 1$$

$$D'' = S'' * \int \frac{C}{\rho} d\tilde{s} - C'' * \int \frac{S}{\rho} d\tilde{s} + \frac{1}{\rho}$$

remember: S & C are solutions of the homog. equation of motion:

$$S'' + K * S = 0$$
$$C'' + K * C = 0$$

$$D'' = -K * S * \int \frac{C}{\rho} d\tilde{s} + K * C * \int \frac{S}{\rho} d\tilde{s} + \frac{1}{\rho}$$

$$D'' = -K * \underbrace{\left\{ S \int \frac{C}{\rho} d\tilde{s} + C \int \frac{S}{\rho} d\tilde{s} \right\}}_{=D(s)} + \frac{1}{\rho}$$

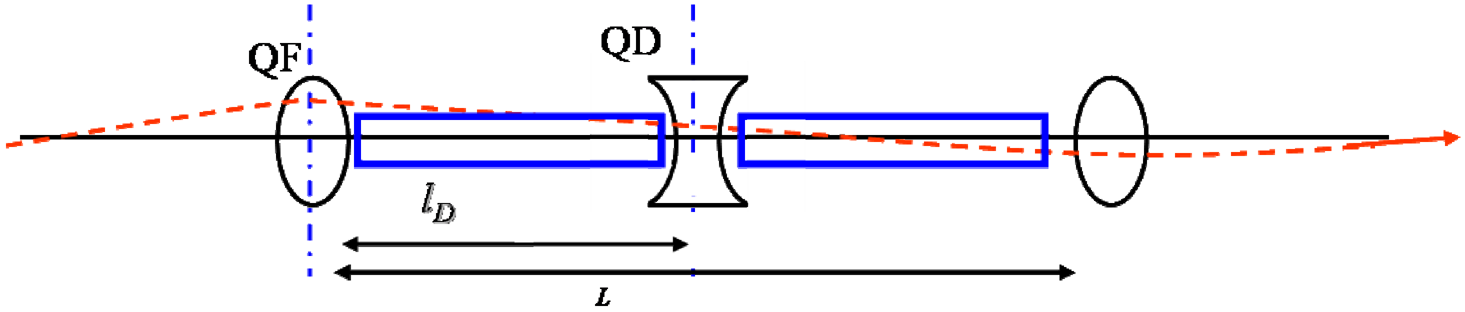
$$D'' = -K * D + \frac{1}{\rho}$$

… or

$$D'' + K * D = \frac{1}{\rho}$$

qed

Dispersion in a FoDo Cell:



!! we have now introduced dipole magnets in the FoDo:

→ ***we still neglect the weak focusing contribution $1/\rho^2$***

→ ***but take into account $1/\rho$ for the dispersion effect***

assume: length of the dipole = l_D

Calculate the matrix of the FoDo half cell in thin lens approximation:

in analogy to the derivations of $\hat{\beta}, \check{\beta}$

**** thin lens approximation:*** $$f = \frac{1}{k\ell_Q} >> \ell_Q$$

**** length of quad negligible*** $$\ell_Q \approx 0, \rightarrow \ell_D = \frac{1}{2}L$$

**** start at half quadrupole*** $$\frac{1}{\tilde{f}} = \frac{1}{2f}$$

Matrix of the half cell

$$M_{HalfCell} = M_{\frac{QD}{2}} * M_B * M_{\frac{QF}{2}}$$

$$M_{Half\ Cell} = \begin{pmatrix} 1 & 0 \\ \frac{1}{\tilde{f}} & 1 \end{pmatrix} * \begin{pmatrix} 1 & \ell \\ 0 & 1 \end{pmatrix} * \begin{pmatrix} 1 & 0 \\ \frac{-1}{\tilde{f}} & 1 \end{pmatrix}$$

$$M_{Half\ Cell} = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} = \begin{pmatrix} 1 - \frac{\ell}{\tilde{f}} & \ell \\ \frac{-\ell}{\tilde{f}^2} & 1 + \frac{\ell}{\tilde{f}} \end{pmatrix}$$

calculate the dispersion terms D, D´ from the matrix elements

$$\boldsymbol{D(s) = S(s) * \int \frac{1}{\rho(\tilde{s})} C(\tilde{s}) d\tilde{s} - C(s) * \int \frac{1}{\rho(\tilde{s})} S(\tilde{s}) d\tilde{s}}$$

$$D(\ell) = \ell * \frac{1}{\rho} * \int_0^{\ell} \left(1 - \frac{s}{\tilde{f}}\right) ds - \left(1 - \frac{\ell}{\tilde{f}}\right) * \frac{1}{\rho} * \int_0^{\ell} s \quad ds$$

S(s) C(s) C(s) S(s)

$$D(\ell) = \frac{\ell}{\rho}\left(\ell - \frac{\ell^2}{2\tilde{f}}\right) - \left(1 - \frac{\ell}{\tilde{f}}\right) * \frac{1}{\rho} * \frac{\ell^2}{2} = \frac{\ell^2}{\rho} - \frac{\ell^3}{2\tilde{f}\rho} - \frac{\ell^2}{2\rho} + \frac{\ell^3}{2\tilde{f}\rho}$$

$$D(\ell) = \frac{\ell^2}{2\rho}$$

in full analogy on derives for D´:

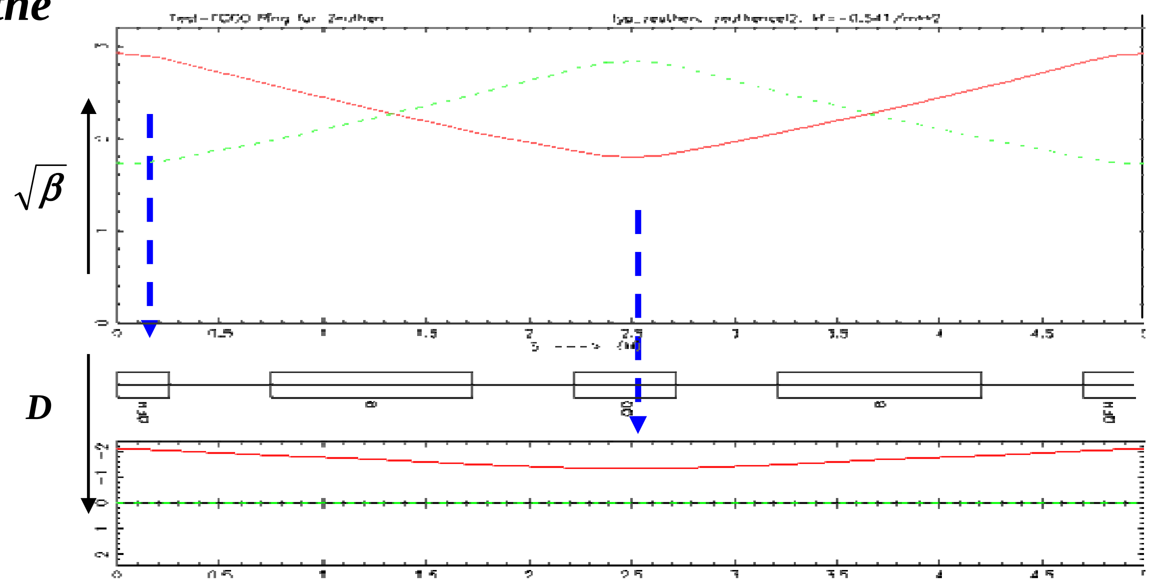
$$D'(s) = \frac{\ell}{\rho}\left(1 + \frac{\ell}{2\tilde{f}}\right)$$

and we get the complete matrix including the dispersion terms D, D´

$$M_{halfCell} = \begin{pmatrix} C & S & D \\ C' & S' & D' \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1-\frac{\ell}{\tilde{f}} & \ell & \frac{\ell^2}{2\rho} \\ \frac{-\ell}{\tilde{f}^2} & 1+\frac{\ell}{\tilde{f}} & \frac{\ell}{\rho}(1+\frac{\ell}{2\tilde{f}}) \\ 0 & 0 & 1 \end{pmatrix}$$

boundary conditions for the transfer from the center of the foc. to the center of the defoc. quadrupole

$$\begin{pmatrix} \check{D} \\ 0 \\ 1 \end{pmatrix} = M_{1/2} * \begin{pmatrix} \hat{D} \\ 0 \\ 1 \end{pmatrix}$$



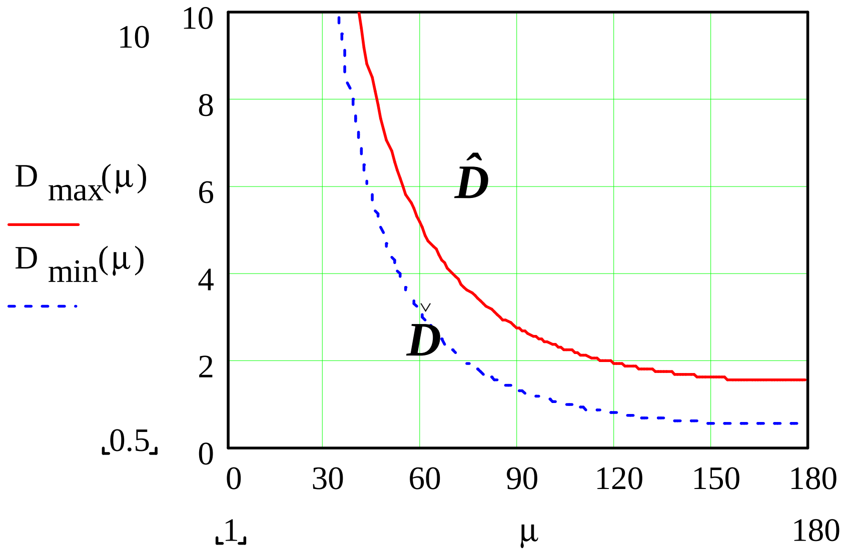
Dispersion in a FoDo Cell

$$\rightarrow \check{D} = \hat{D}\left(1-\frac{\ell}{\tilde{f}}\right)+\frac{\ell^2}{2\rho}$$

$$\rightarrow 0 = -\frac{\ell}{\tilde{f}^2} * \hat{D} + \frac{\ell}{\rho}\left(1+\frac{\ell}{2\tilde{f}}\right)$$

$$\hat{D} = \frac{\ell^2}{\rho} * \frac{\left(1+\frac{1}{2}\sin\frac{\mu}{2}\right)}{\sin^2\frac{\mu}{2}} \qquad \check{D} = \frac{\ell^2}{\rho} * \frac{\left(1-\frac{1}{2}\sin\frac{\mu}{2}\right)}{\sin^2\frac{\mu}{2}}$$

where μ denotes the phase advance of the full cell and $l/f = \sin(\mu/2)$



Nota bene:

***!** small dispersion needs strong focusing*
→ large phase advance

***!!** ↔ there is an optimum phase for small β*

***!!!** ...do you remember the stability criterion?*
½ trace = cos μ ↔ μ < 180°

***!!!!** … life is not easy*

Measuring the Dispersion

Idea: ***apply a well defined momentum shift Δp /p of the beam***
without changing the magnetic fields } α_{cp}

revolution time

$$T_0 = \frac{L_0}{v} \quad \rightarrow \quad \frac{dT_0}{T_0} = \frac{dL_0}{L_0} - \frac{dv}{v}$$

$$\frac{dT_0}{T_0} = \alpha_{cp}\frac{dp}{p} - \frac{dv}{v} \qquad \Bigg| \qquad \alpha_{cp} = \frac{dL}{L} / \frac{dp}{p}$$

$$-\frac{df}{f} = \alpha_{cp}\frac{dp}{p} - \frac{1}{\gamma^2}\frac{dp}{p} \qquad \Bigg| \qquad \frac{dv}{v} = \frac{1}{\gamma^2} * \frac{dp}{p}$$

$$-\frac{df}{f} = \frac{dp}{p}(\alpha_{cp} - \frac{1}{\gamma^2})$$

Measuring the Dispersion

$$\frac{df}{f} = \frac{dp}{p}\left(\underbrace{\frac{1}{\gamma^2}}_{\approx 0} - \alpha_{cp}\right)$$

≈ 0 *for high energy beams.*

Example:

$$df = 360 Hz$$
$$f = 208 MHz$$
$$\alpha_{cp} = 1.2 * 10^{-3}$$

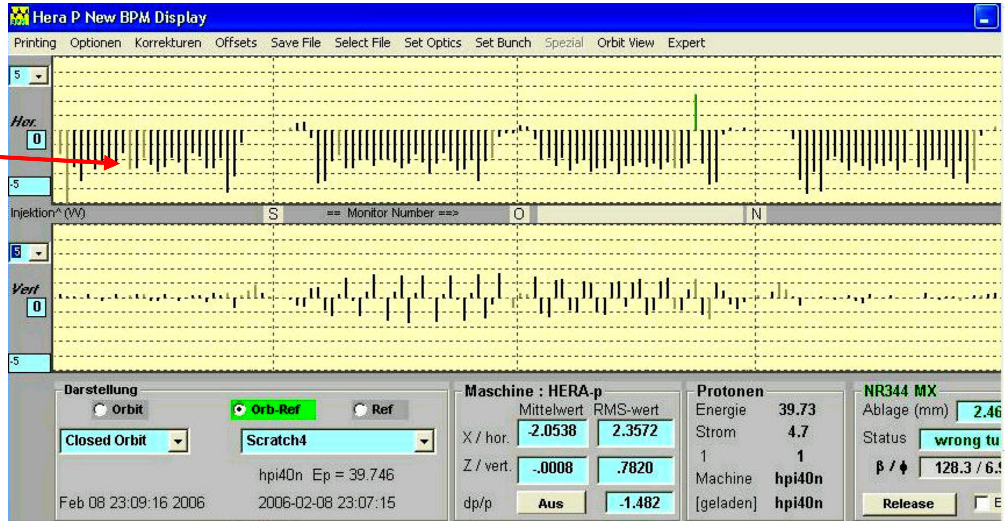
$$\frac{dp}{p} = -\frac{df}{f} / \alpha_{cp} = \frac{360 Hz}{208 MHz * 1.2 * 10^{-3}}$$

$$\frac{dp}{p} = 1.4 * 10^{-3}$$

Dispersion Function in the arc:

$$\hat{D} = 2\,m$$

$$\rightarrow \quad x_D \approx 3\,mm$$



HERA Dispersion Orbit