Insertion Devices
Lecture 2
Wigglers and Undulators

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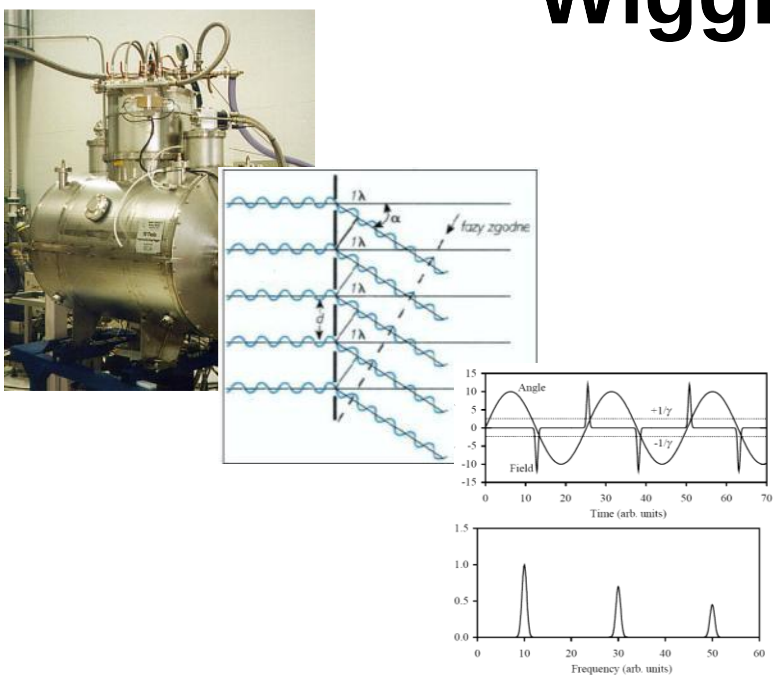


Summary from Lecture #1

Synchrotron Radiation is emitted by accelerated charged particles

The combination of Lorentz contraction and the Doppler shift turns the **cm length scale into nm wavelengths** (making SR the best possible source of X-rays)

Apply Maxwell's equations to the particle, taking care to relate the emitted time to the observed time

Bending magnet radiation is characterised by a critical frequency

Bending Magnet Brightness (or sometimes "Brilliance")

All emitted photons have a position and an angle in phase space (x, x')

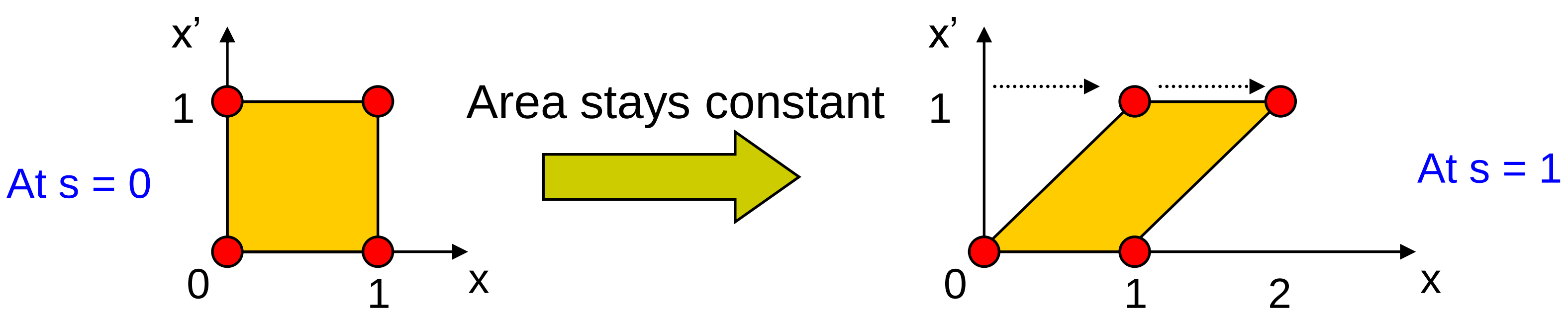
Phase space evolves as photons travel but the area stays constant (Liouville's theorem)

The emittance of an electron beam is governed by the same theorem

Brightness is the phase space density of the flux – takes account of the number of photons and their concentration

Brightness (like flux) is conserved by an ideal optical transport system, unlike angular flux density for instance

Since it is conserved it is a **good figure of merit** for comparing sources (like electron beam emittance)



Brightness

To calculate the brightness we need the phase space areas

We need to include the photon and electron contributions

The horizontal angle is considered separately since light is emitted smoothly over the full 2π

The **effective** vertical angle is $\Sigma_{y'} = \sqrt{\sigma_{y'}^2 + \sigma_{r'}^2}$

We add contributions in quadrature as both are assumed to be Gaussian distributions

The horizontal and vertical effective sizes are similarly:

$$\Sigma_x = \sqrt{\sigma_x^2 + \sigma_r^2}$$

$$\Sigma_y = \sqrt{\sigma_y^2 + \sigma_r^2}$$

Brightness

The photon beam size is found by assuming the source is the fundamental mode of an optical resonator (called the Gaussian laser mode)

$$2\pi\sigma_r\sigma_{r'} = \frac{\lambda}{2}$$

Bending magnet brightness is then

$$B = \frac{\dot{N}}{(2\pi)^{3/2}\Sigma_x\Sigma_y\Sigma_{y'}}$$

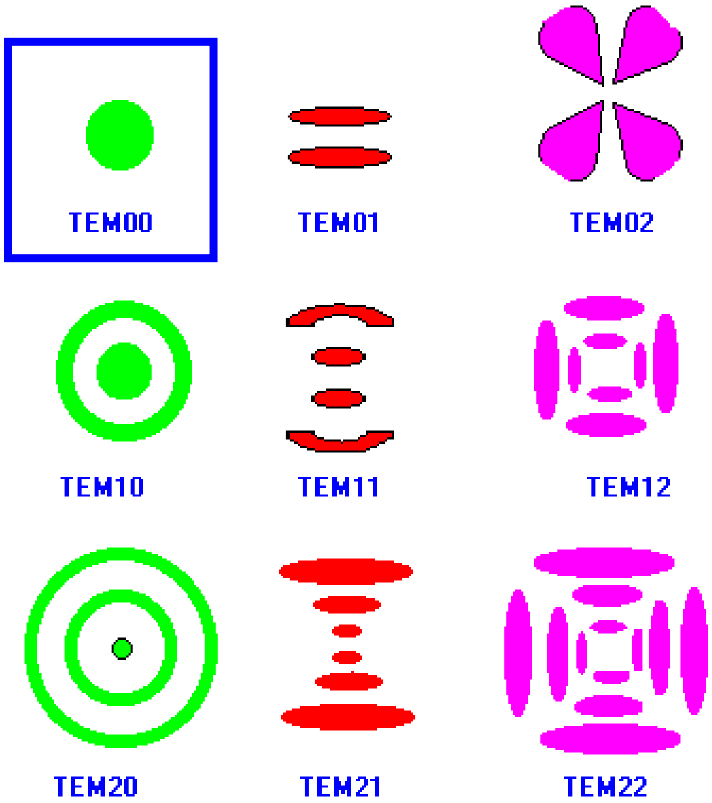
Each term contributes $\sqrt{2\pi}$ because the rectangular function of equal area to a Gaussian has width $\sqrt{2\pi}\sigma$

In general $\sigma_r \ll \sigma_{x,y}$ and so

$$B = \frac{\dot{N}}{(2\pi)^{3/2}\sigma_x\sigma_y(\sigma_{y'}^2 + \sigma_{r'}^2)^{1/2}}$$

The units are photons/s/solid area/solid angle/spectral bandwidth

Resonator Modes



Transverse Electromagnetic Modes – possible standing waves within a laser cavity

Power

Virtually all SR facilities have melted vacuum chambers or other components due to the SR hitting an uncooled surface

The average power is high but the **power density is very high** – the power is concentrated in a tight beam. The total power emitted by an electron beam in 360° of bending magnets is

$$\boxed{P_{\mathrm{total}} = 88.46\,\frac{E^4 I_b}{\rho_0}}$$

where the power is in kW, E is in GeV, I_b is in A, ρ_0 is in m.

Other useful values are the power per horizontal angle (in W/mrad) and power density on axis (in W/mrad^2)

$$\frac{dP}{d\theta} = 14.08\,\frac{E^4 I_b}{\rho_0} \qquad \left.\frac{dP}{d\Omega}\right|_{\psi=0} = 18.08\,\frac{E^5 I_b}{\rho_0}$$

Examples

Ring	Energy (GeV)	ρ (m)	I_b (mA)	P_{total} (kW)	$dP/d\theta$ (W/mrad)	$dP/d\Omega$ (W/mrad^2)
SRS	2	5.56	200	50.9	8.1	20.8
DIAMOND	3	7.15	300	300.7	47.9	184.4
ESRF	6	25.0	200	916.5	145.9	1124.0

Bending Magnet Spectrum

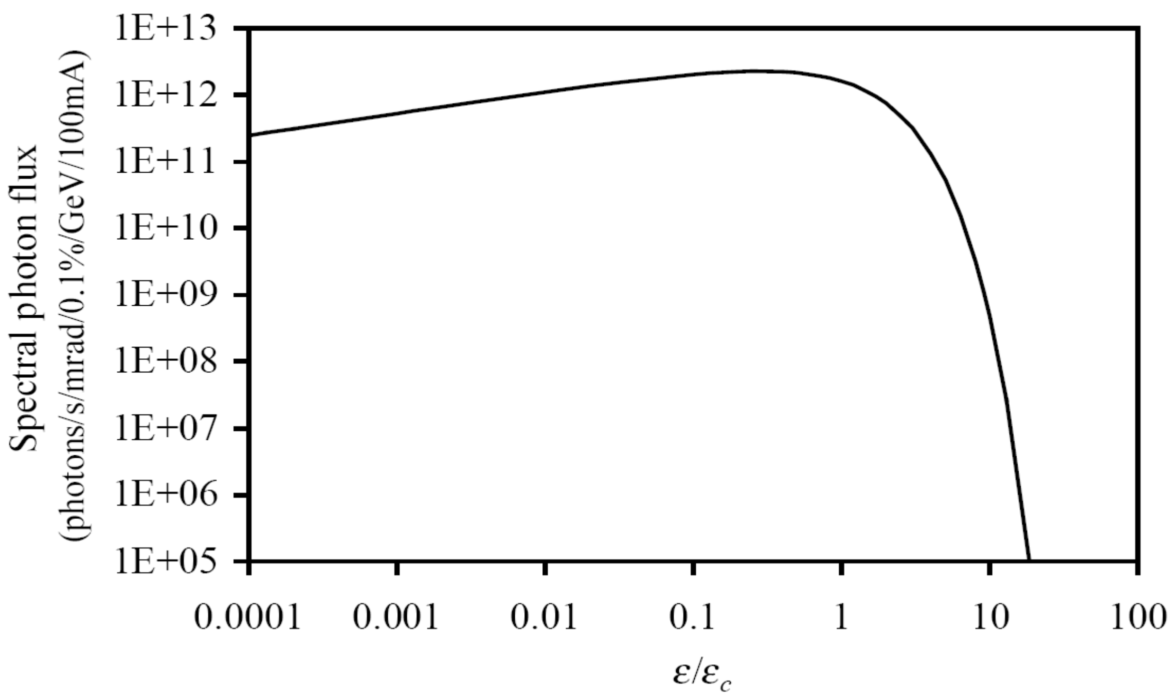
A plot of $\dot{N}$ versus ϵ/ϵ_c gives the **universal curve**

All flux plots from bending magnets have the same characteristic shape

Once the **critical energy is known** it is easy to find (scale off) the photon flux

The amplitude changes with E and I_b so higher energies and higher beam currents give more flux

Note the log-log scale



Bending Magnet Spectrum

In a storage ring of fixed energy, the spectrum can be shifted sideways along the photon energy axis if a different critical energy can be generated.

$$\boxed{\epsilon_c = \hbar\omega_c = \frac{3hc\gamma^3}{4\pi\rho}}$$

Need to change ρ (B Field)

Used especially to shift the rapidly falling edge (high energy photons, short wavelengths)

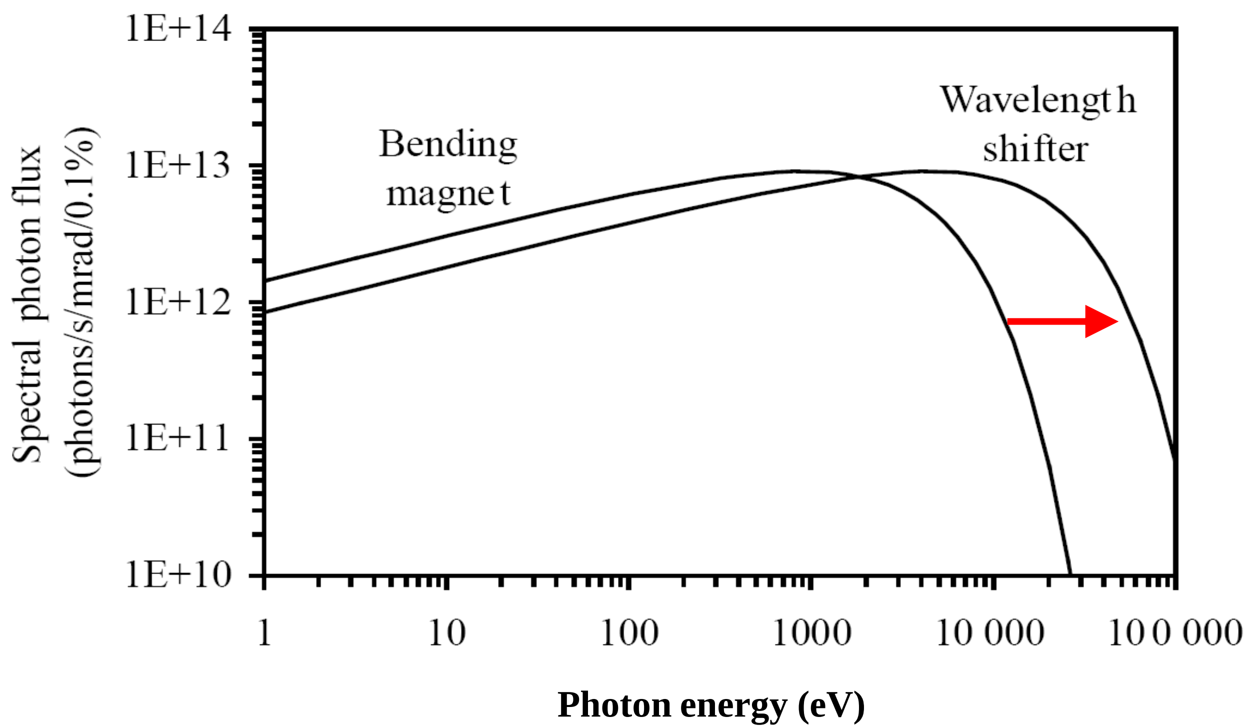
Special magnets that do this are called **wavelength shifters**

An alternative is to replace individual bending magnets with higher strength ones (superbends) – this is not popular but it has been done

Wavelength Shifters (WS)

Shift the critical energy by locally changing the bending magnet field

The shape of the curve is unchanged but the spectrum is shifted



SRS Example

1.2T BM & 6T WS. 2GeV, 200mA.

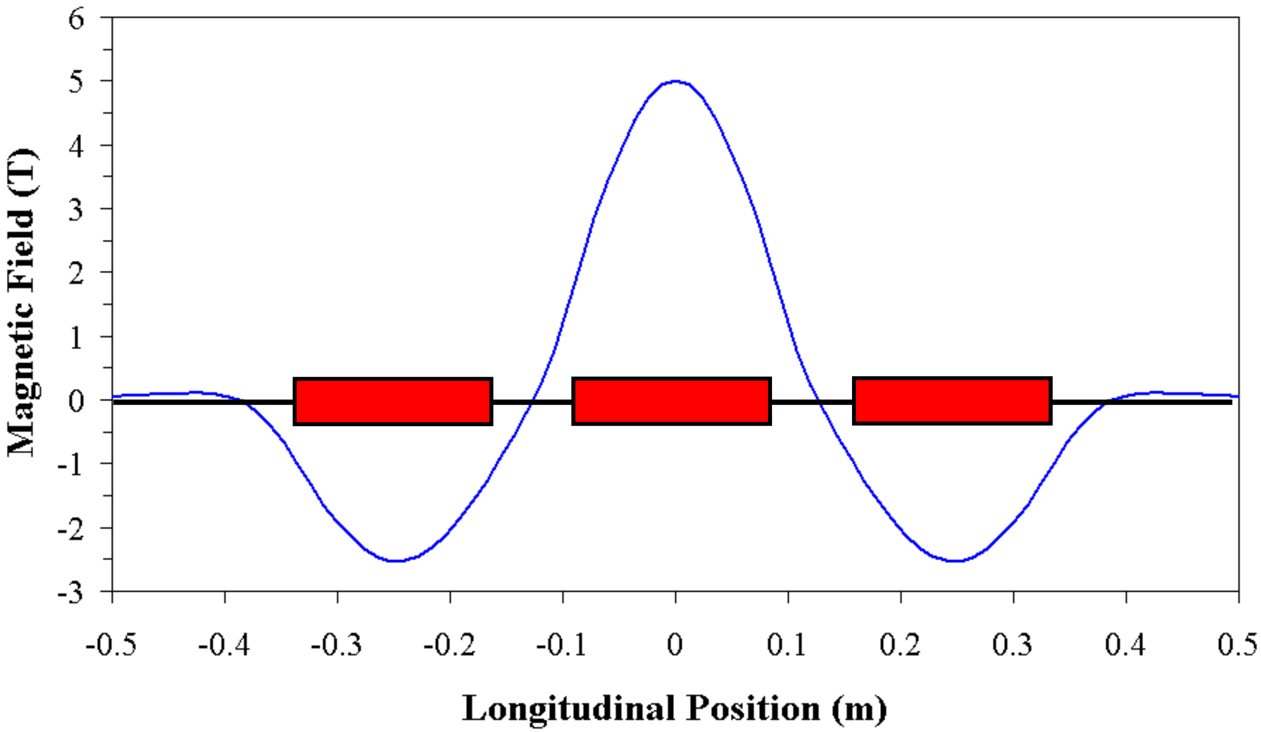
Flux at 30keV increased by x100

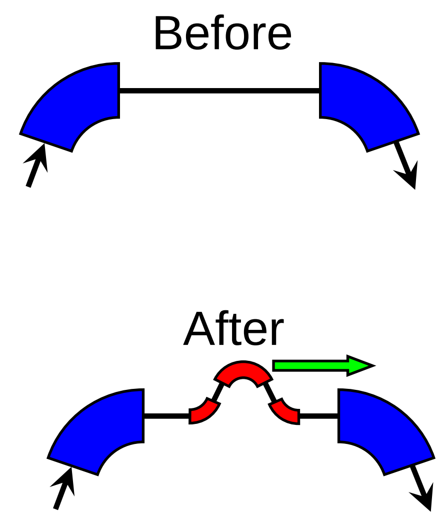
Wavelength Shifters

How can you put a high magnetic field into a ring?

A popular solution is to use 3 magnets to create a chicane like trajectory on the electron beam in a straight section

The central magnet is the **high field** bending magnet source



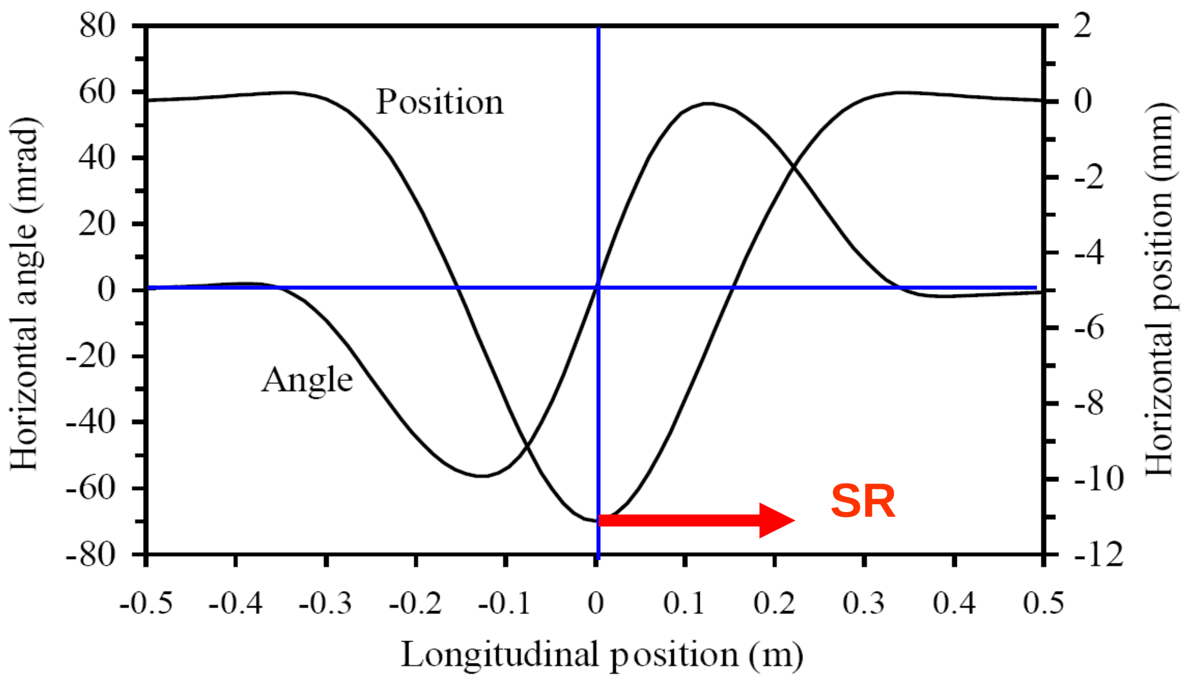


Electron trajectory in a Wavelength Shifter

The electron enters on axis and exits on axis **("Insertion Device")**

The peak of the trajectory bump occurs at the peak magnetic field – when the angle is zero

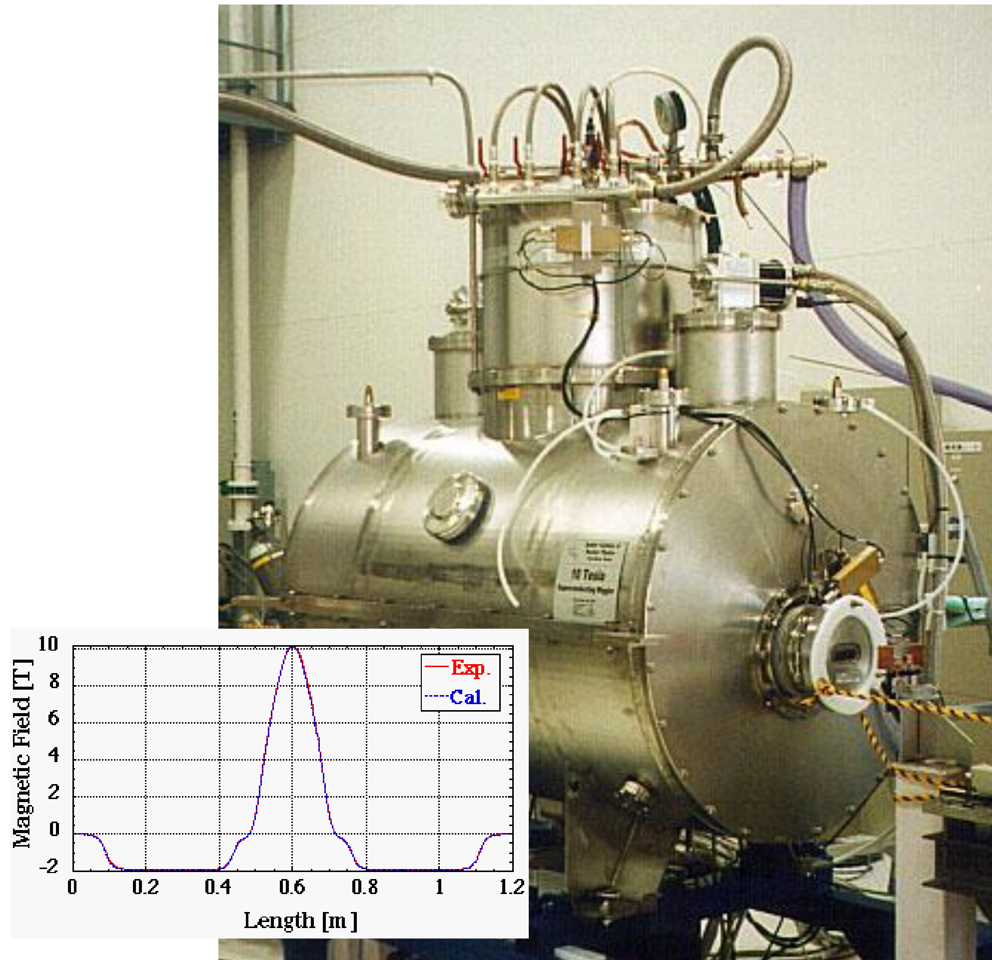
SR emitted here will travel parallel to the beam axis (at a tangent to the trajectory)



Examples of Wavelength Shifters

SRS 6T (central pole) wavelength shifter

Spring-8 10T wiggler



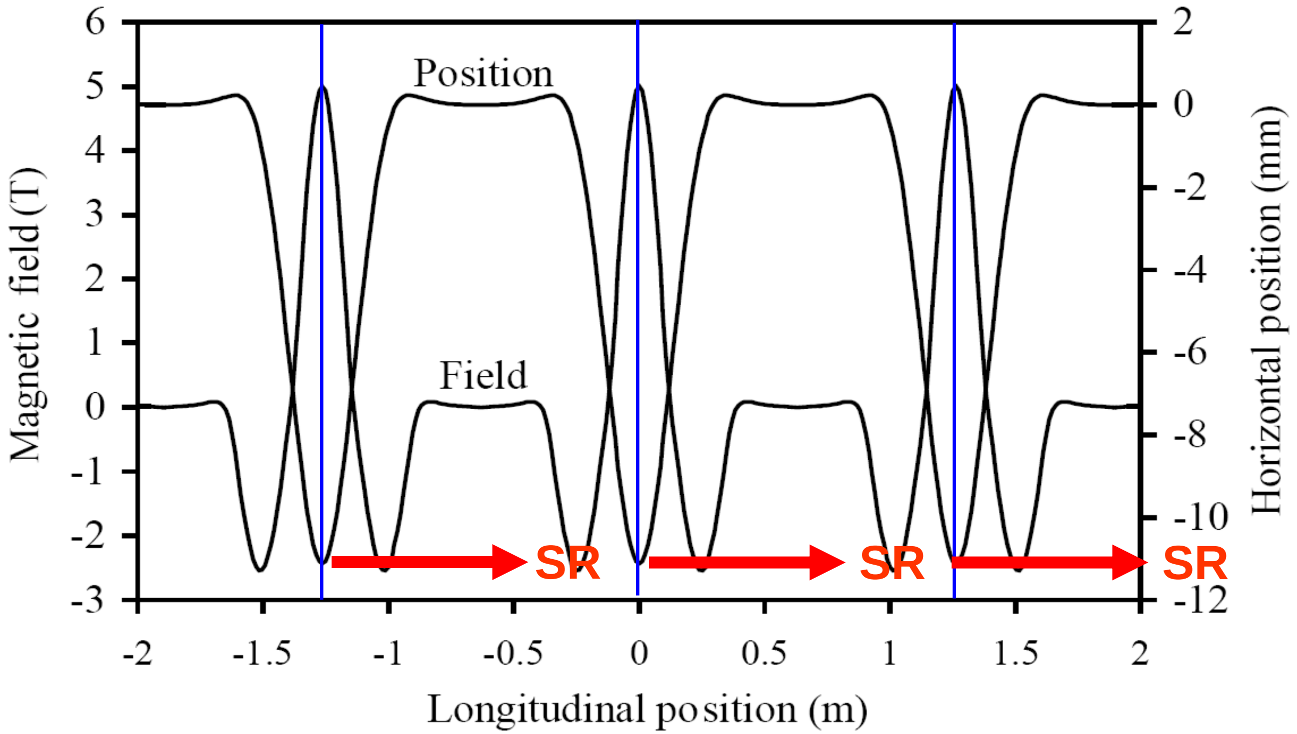
Wavelength shifters are always superconducting magnets

Extension to Multipole Wigglers

One wavelength shifter will give enhanced flux at high photon energies

SR is emitted parallel to the axis at the peak of the main pole

Imagine many WS installed next to each other in the same straight …



Multiple Wavelength Shifters

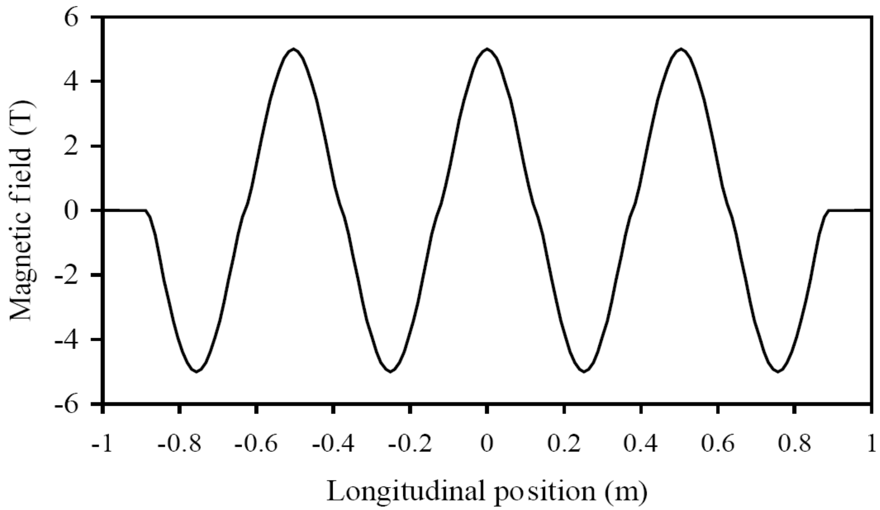
Each WS would be an independent source of SR – all emitting in the forward direction.

The observer (on-axis) would see SR from all 3 Source points

The observer will therefore see 3 times more flux

This is the basic concept for a **multipole wiggler**

Three separate WS is not the most efficient use of the space! A better way of packing more high field emitters into a straight is…



B field is usually close to sinusoidal

Multipole Wigglers – Electron Trajectory

Electrons travelling in the **s** direction

Assuming small angular deflections $(\dot{x} \ll 1, \dot{y} \ll 1)$

The equations of motion for the electron are

$$\ddot{x} = \frac{d^2x}{ds^2} = \frac{e}{\gamma m_0 c}(B_y - \dot{y}B_s)$$

$$\ddot{y} = \frac{d^2y}{ds^2} = \frac{e}{\gamma m_0 c}(\dot{x}B_s - B_x)$$

If we have a MPW which only deflects in the horizontal plane (x) - only has vertical fields (B_y) on axis

$$\ddot{x} = \frac{eB_y}{\gamma m_0 c}$$

$$\ddot{y} = 0 \ .$$

Angular Deflection

The B field is assumed to be **sinusoidal** with period λ_u

$$B_y(s) = -B_0 \sin\left(\frac{2\pi s}{\lambda_u}\right)$$

Integrate once to find $\dot{x}$ which is the horizontal angular deflection from the s axis

$$\dot{x}(s) = \frac{B_0 e}{\gamma m_0 c}\frac{\lambda_u}{2\pi}\cos\left(\frac{2\pi s}{\lambda_u}\right)$$

Therefore, the peak angular deflection is $\frac{B_0 e}{\gamma m_0 c}\frac{\lambda_u}{2\pi}$

Define the **deflection parameter**

$$\boxed{K = \frac{B_0 e}{m_0 c}\frac{\lambda_u}{2\pi} = 93.36\, B_0 \lambda_u}$$

(B_0 in T, λ_u in m)

Trajectory

One more integration gives

$$x(s) = \frac{K}{\gamma}\frac{\lambda_u}{2\pi}\sin\left(\frac{2\pi s}{\lambda_u}\right)$$

The peak angular deflection is $\frac{K}{\gamma}$

Remember that SR is emitted with a typical angle o $\sim 1/\gamma$

So if $K < 1$ the electron trajectory will overlap with the emitted cone of SR (**an undulator**)

If $K \gg 1$ there will be little overlap and the source points are effectively independent – this is the case for a **MPW**

The boundary between an undulator and a MPW is not actually so black and white as this!

MPW Flux

Can be considered a series of dipoles, one after the other

There are two source points per period

The flux is simply the product of the number of source points and the dipole flux for that critical energy

The MPW has two clear advantages

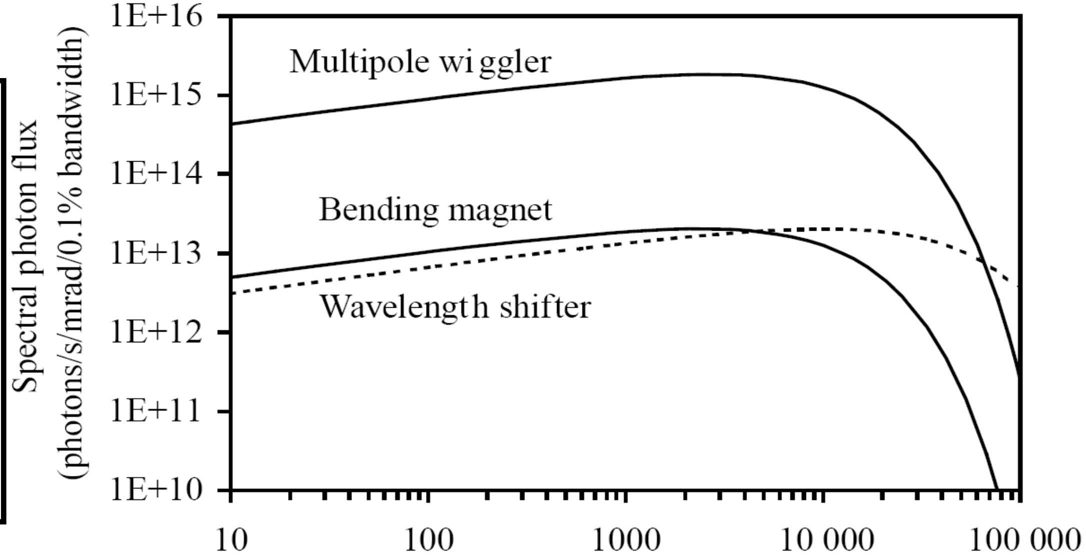
- **The critical energy can be set to suit the science need**
- **The Flux is enhanced by twice number of periods**

300mA, 3 GeV beam

1.4T dipole

6T WS

1.6T, MPW with 45 periods
(2 poles per period so x90 flux)



MPW Power

The total power emitted by a beam of electrons passing through **any magnet system** is

$$\boxed{P_{\text{total}} = 1265.5\, E^2 I_b \int_0^L B(s)^2 ds}$$

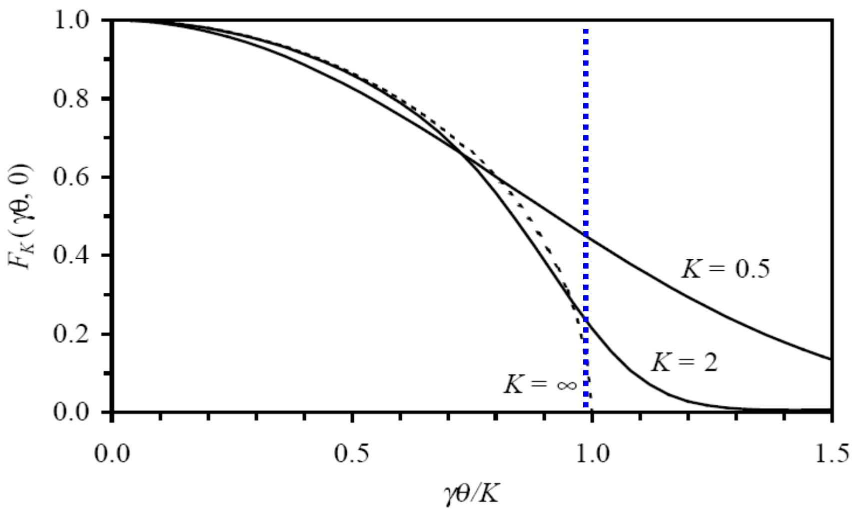
This is a general result – can get the earlier bending magnet result from here.

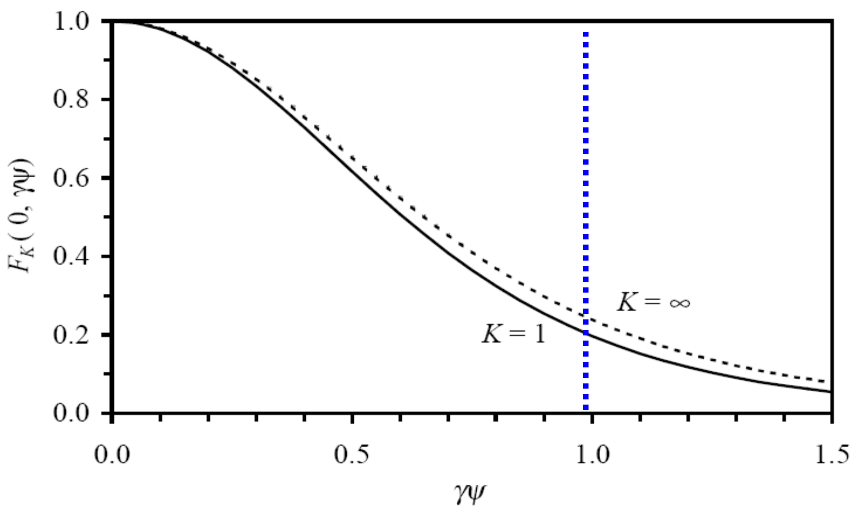
For a sinusoidal magnetic field with peak value B_0 the integral is $B_0^2 L/2$ and so the total power emitted is (in W)

$$\boxed{P_{\text{total}} = 632.8\, E^2 B_0^2 L I_b}$$

Power Density

The power is contained in **K/γ** horizontally for large K





Vertically, the power is contained in ~ **1/γ**

On-Axis power density

The **Peak** power density is **on-axis**

	Energy (GeV)	B_0 (T)	K	L (m)	I_b (mA)	P_{total} (kW)	$dP/d\Omega$ (kW/mrad^2)
Undulators	2	0.64	3	5	200	1.0	2.2
	3	0.64	3	5	300	3.5	16.8
	6	0.64	3	5	200	9.3	179.1
MPWs	2	2	40	2	200	4.0	0.6
	3	2	40	2	300	13.7	4.9
	6	2	40	2	200	36.4	52.5

Undulators

For a sinusoidal magnetic field

$$\dot{x}(s) = \frac{dx}{ds} = \frac{K}{\gamma} \cos\left(\frac{2\pi s}{\lambda_u}\right)$$

β_x is the relative transverse velocity

$$\beta_x = \frac{dx/dt}{c} = \frac{K}{\gamma} \cos\left(\frac{2\pi s}{\lambda_u}\right) \qquad \Delta s = c\Delta t$$

The energy is fixed so β is also fixed. Any variation in β_x will have a corresponding change in β_s ($\beta_y = 0$)

$$\begin{aligned} \beta_s^2 &= \beta^2 - \beta_x^2 \\ &= \beta^2 - \frac{K^2}{\gamma^2} \cos^2\left(\frac{2\pi s}{\lambda_u}\right) \end{aligned}$$

The Undulator Equation

Using $\cos 2A = 2\cos^2 A - 1$

$$\beta_s^2 = \beta^2 - \frac{K^2}{\gamma^2}\left(\frac{1}{2} + \frac{1}{2}\cos\left(\frac{4\pi s}{\lambda_u}\right)\right)$$

And then using $(1-x)^{1/2} \sim 1 - x/2$

$$\beta_s \sim \beta\left(1 - \frac{K^2}{4\beta^2\gamma^2} - \frac{K^2}{4\beta^2\gamma^2}\cos\left(\frac{4\pi s}{\lambda_u}\right)\right)$$

This is a constant with an oscillating cosine term. The relative **average** velocity in the forward direction is simply

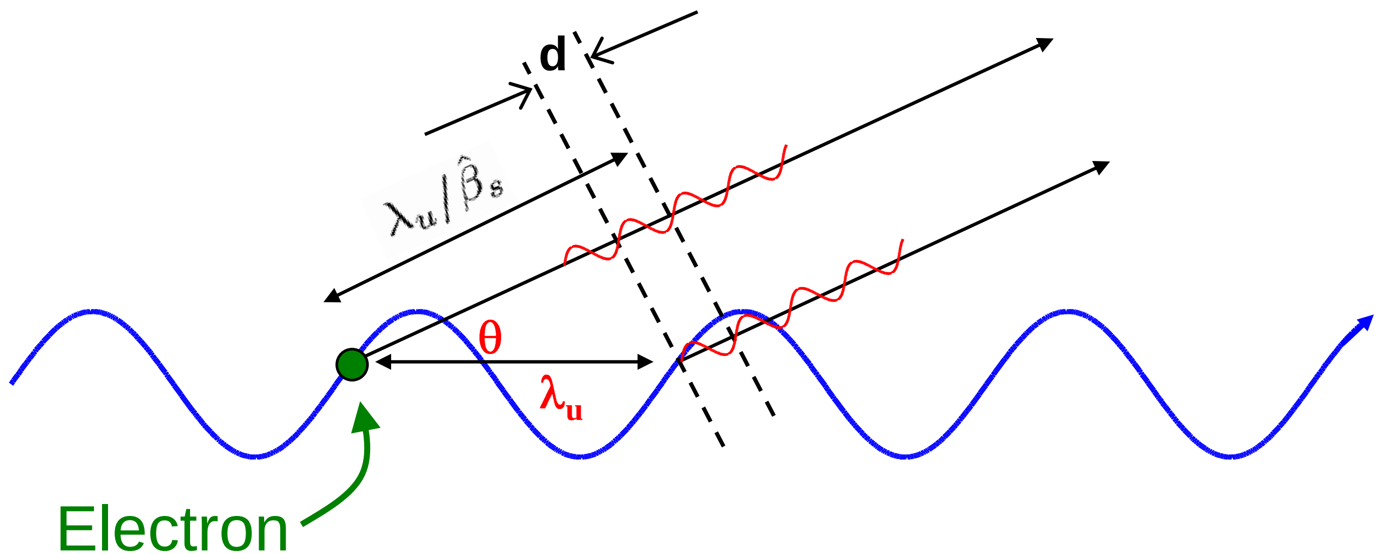
$$\hat{\beta}_s \sim \beta - \frac{K^2}{4\beta\gamma^2}$$

$$\sim 1 - \frac{1}{2\gamma^2} - \frac{K^2}{4\beta\gamma^2}$$

$$\beta = \sqrt{1 - \frac{1}{\gamma^2}}$$

The Condition for Interference

For constructive interference between wavefronts emitted by the same electron **the electron must slip back by a whole number of wavelengths** over one period



Speed = distance/time = c

The time for the electron to travel one period is $\lambda_u/c\hat{\beta}_s$

In this time the first wavefront will travel the distance $\lambda_u/\hat{\beta}_s$

Interference Condition

The separation between the wavefronts is $d = \frac{\lambda_u}{\hat{\beta}_s} - \lambda_u \cos\theta$

And this **must equal a whole number of wavelengths** for constructive interference

$$n\lambda = \frac{\lambda_u}{\hat{\beta}_s} - \lambda_u \cos\theta$$

n is an integer – the harmonic number

Using $(1-x)^{-1} \sim 1 + x$

We have

$$n\lambda \sim \lambda_u \left(1 + \frac{1}{2\gamma^2} + \frac{K^2}{4\beta\gamma^2}\right) - \lambda_u \cos\theta$$

Using $1 - \cos\theta = 2\sin^2(\theta/2)$ and the small angle approximation $\sin\theta \sim \theta$ ......

Interference Condition

We get

$$n\lambda \sim \frac{\lambda_u \theta^2}{2} + \frac{\lambda_u}{2\gamma^2} + \frac{\lambda_u K^2}{4\beta\gamma^2}$$

$$\sim \frac{\lambda_u}{2\gamma^2}\left(1 + \frac{K^2}{2} + \theta^2\gamma^2\right)$$

And the undulator equation

$$\boxed{\lambda = \frac{\lambda_u}{2n\gamma^2}\left(1 + \frac{K^2}{2} + \theta^2\gamma^2\right)}$$

Example, 3GeV electron passing through a 50mm period undulator with K = 3. First harmonic (n = 1), on-axis is ~4 nm.

cm periods translate to **nm** wavelengths because of the huge γ^2 term

Undulator equation implications

$$\boxed{\lambda = \frac{\lambda_u}{2n\gamma^2}\left(1 + \frac{K^2}{2} + \theta^2\gamma^2\right)}$$

The wavelength primarily depends on the **period** and the **energy** but also on K and the observation angle θ.

$$\boxed{K = \frac{B_0 e}{m_0 c}\frac{\lambda_u}{2\pi} = 93.36\, B_0 \lambda_u}$$

If we change B we can change λ. For this reason, undulators are built with smoothly adjustable B field. The amount of the adjustability sets the tuning range of the undulator.

Note. What happens to λ as B increases?

Undulator equation implications

As **B increases** (and so K), the output **wavelength increases** (photon energy decreases).

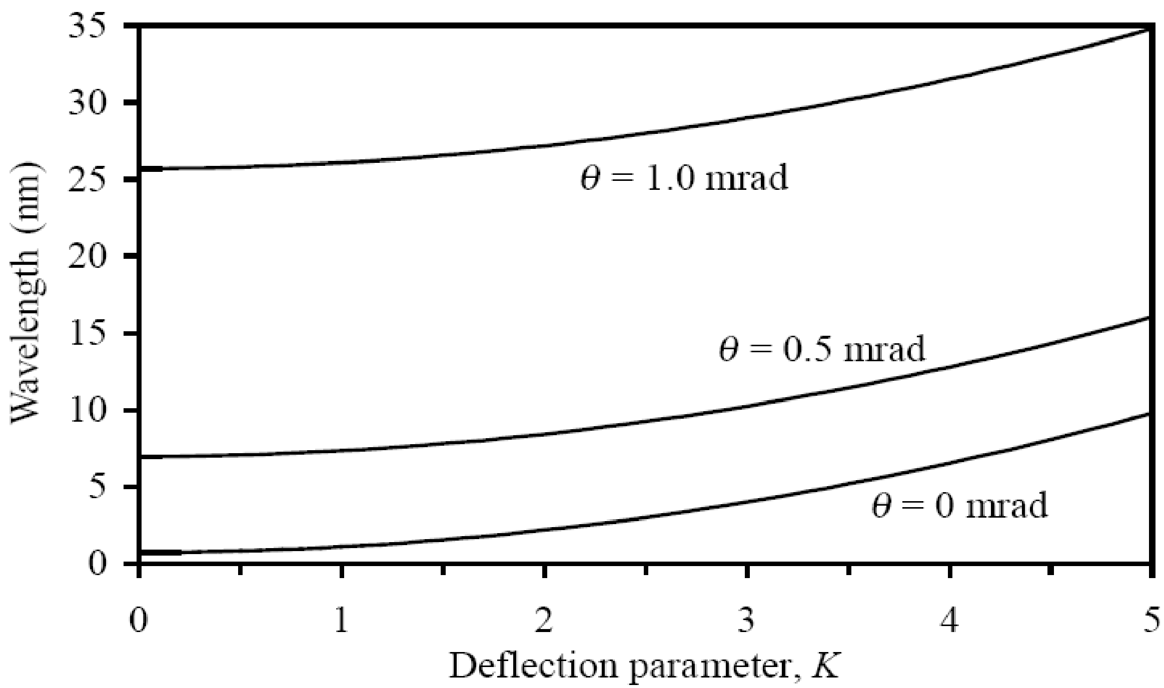
This *appears* different to bending magnets and wigglers where we increase B so as to produce shorter wavelengths (higher photon energies).

The wavelength changes with θ^2, so it always gets longer as you move away from the axis

An important consequence of this is that the **beamline aperture choice is important** because it alters the radiation characteristics reaching the observer.

Example

3GeV electron passing through a 50mm period undulator with K = 3. First harmonic (n = 1), on-axis is ~4 nm.



Note: the wavelength can be varied quite significantly but this graph does not say **how many photons** you will observe!

Harmonic bandwidth

Assume the undulator contains **N periods**

For constructive interference

$$Nn\lambda = \frac{N\lambda_u}{\hat{\beta}_s} - N\lambda_u \cos\theta$$

For destructive interference to first occur

$$Nn\lambda^* + \lambda^* = \frac{N\lambda_u}{\hat{\beta}_s} - N\lambda_u \cos\theta$$

(ray from first source point exactly out of phase with centre one, ray from 2nd source point out of phase with centre+1, etc)

$$Nn\lambda = Nn\lambda^* + \lambda^*$$

Range over which there is some emission $\Delta\lambda = \lambda - \lambda^*$

Bandwidth (width of harmonic line):

$$\boxed{\frac{\Delta\lambda}{\lambda} \sim \frac{1}{Nn}}$$

Angular width

Destructive interference will first occur when

$$Nn\lambda + \lambda = \frac{N\lambda_u}{\hat{\beta}_s} - N\lambda_u \cos\theta^*$$

This gives $N\lambda_u \cos\theta^* + \lambda = N\lambda_u \cos\theta$

And using $\cos\theta \sim 1 - \theta^2/2$

We find, for the radiation emitted on-axis, the intensity falls to zero at

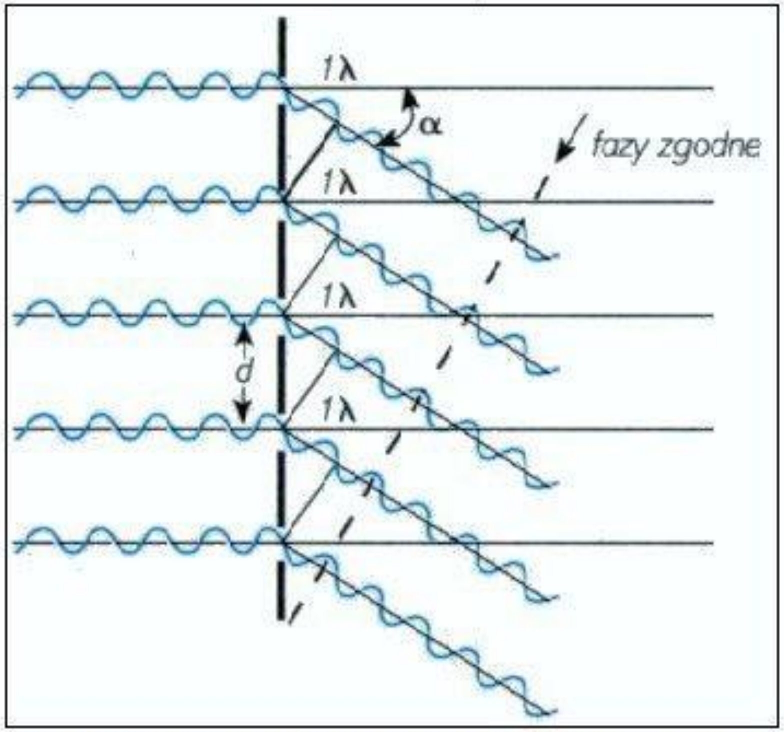
$$\Delta\theta = \sqrt{\frac{2\lambda}{N\lambda_u}}$$

Example, 50mm period undulator with 100 periods emitting 4nm will have $\Delta\theta =$ **40 μrad**, significantly less than $1/\gamma \sim 170\ \mu\text{rad}$

Diffraction Gratings

Very similar results for **angular width** and **bandwidth** apply to **diffraction gratings**

This is because the diffraction grating acts as a large number of equally spaced sources – **very similar concept as an undulator** (but no relativistic effects!)



The diffraction grating and spectrum on screen
d grating constant, λ wave length, α angle of deflection,

Odd and Even Harmonics

There is an important difference in undulators between odd ($n = 1, 3, 5, \ldots$) and even ($n = 2, 4, 6, \ldots$) harmonics

On axis, only odd harmonics are observed

Away from the axis, even harmonics are observed but their characteristics are different (poorer usually)

We will now consider why that might be (simple approach!)

Insertion Device: K << 1

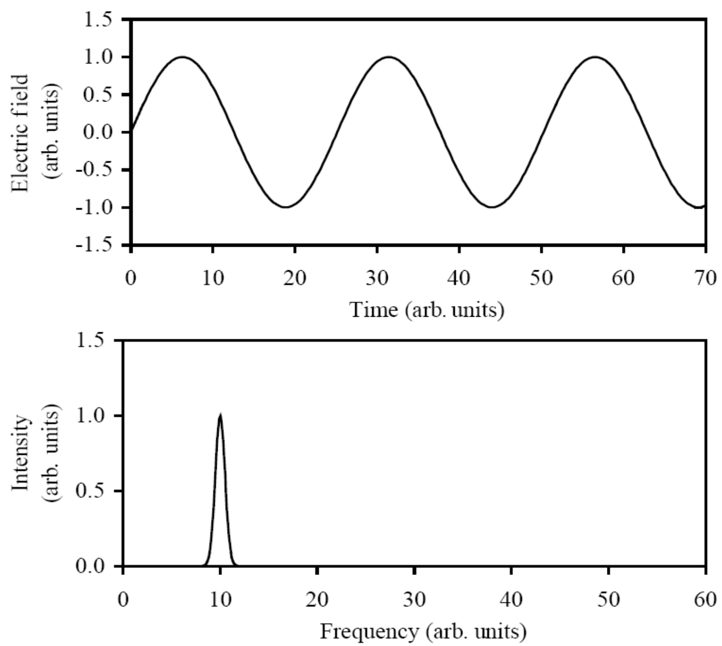
SR is emitted in a cone of $\sim 1/\gamma$

The angular excursion of the electron is K/γ

So the observer **sees all** the emitted radiation

The electric field experienced is a **continuous sinusoidal one**

Fourier analysis of this shows it will be **a single frequency**, so there will be a single harmonic (n = 1)



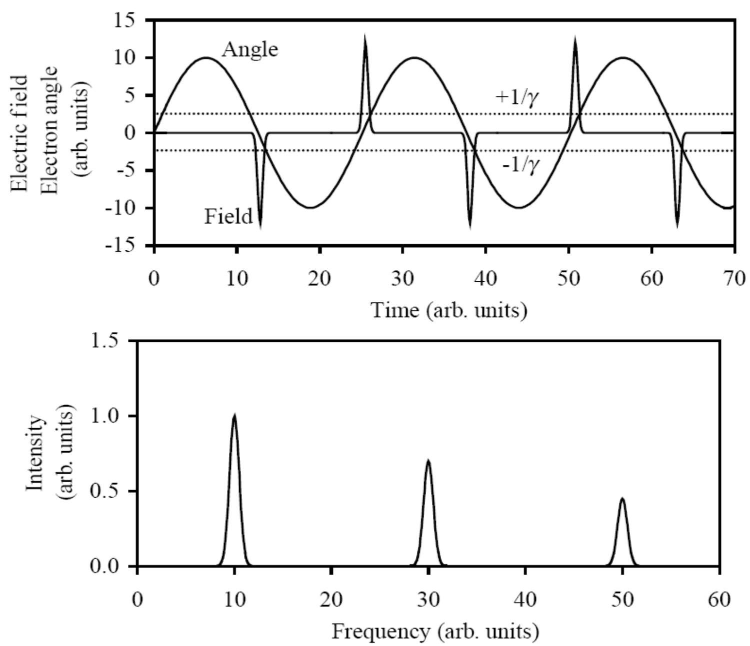
Insertion Device : K >> 1

The observer only experiences an electric field when the electron emission cone flashes past

On axis, the electric field peaks **are equally spaced**

The Fourier Transform of evenly spaced alternating peaks only contains odd harmonics

The sharpness of the electric field spikes increases as K increases so the radiation spectrum contains higher and higher frequencies (higher harmonics)



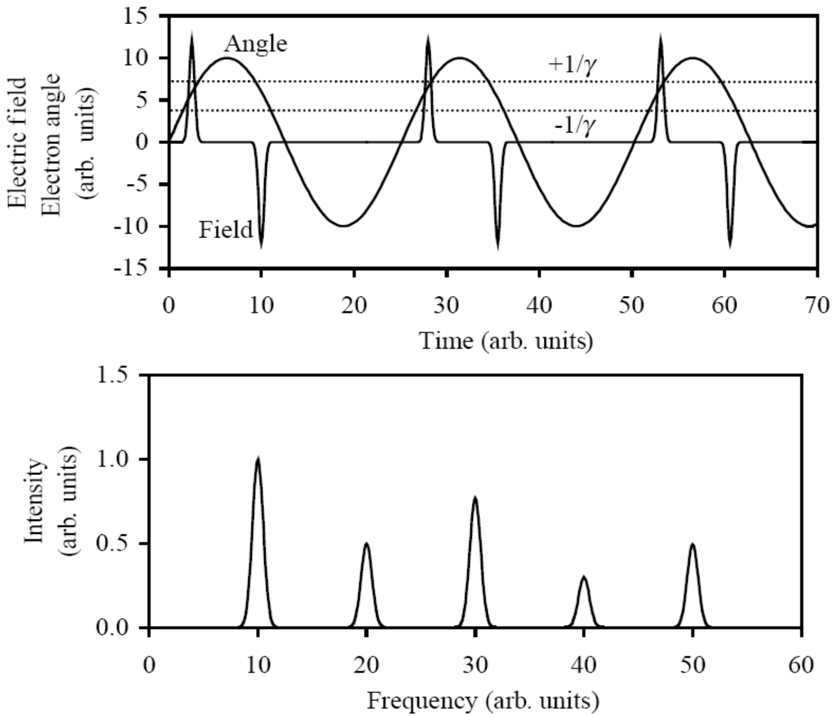
Observation Range

Insertion Device : K >> 1 off axis

Still only see flashes of electric field

No longer evenly spaced

Fourier Transform has to contain even harmonics



Observation Range

When does an undulator become a wiggler?

As K increases the number of harmonics increases:

K	Number of Harmonics
1	few
5	10s
10	100s
20	1000s

At high frequencies the **spectrum smoothes out** and takes on the form of the **bending magnet spectrum**

At low frequencies distinct harmonics are still present and visible

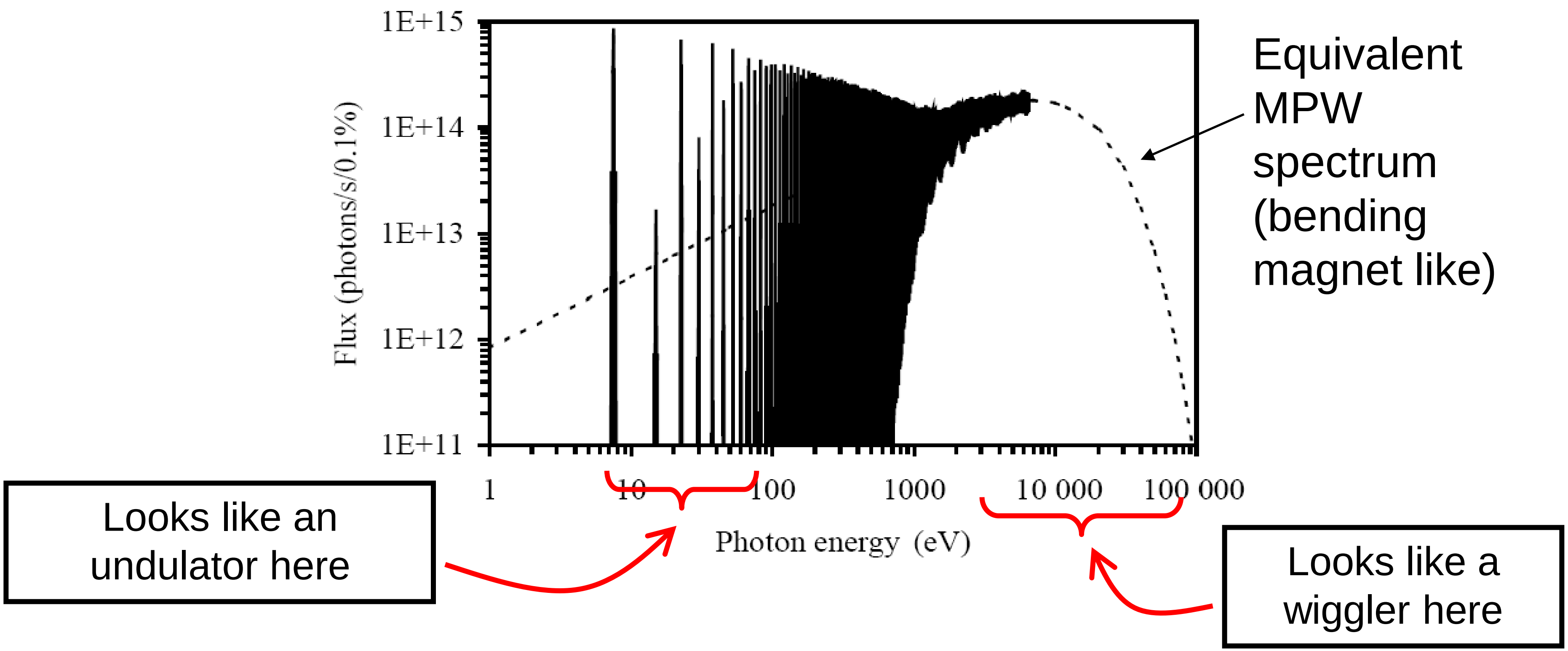
There is in fact **no clear distinction** between an undulator and a wiggler – ***it depends which bit of the spectrum you are observing***

Undulator or Wiggler?

The difference depends upon which bit of the spectrum you use!

This example shows an undulator calculation for K = 15.

[Calculation truncated at high energies as too slow!]



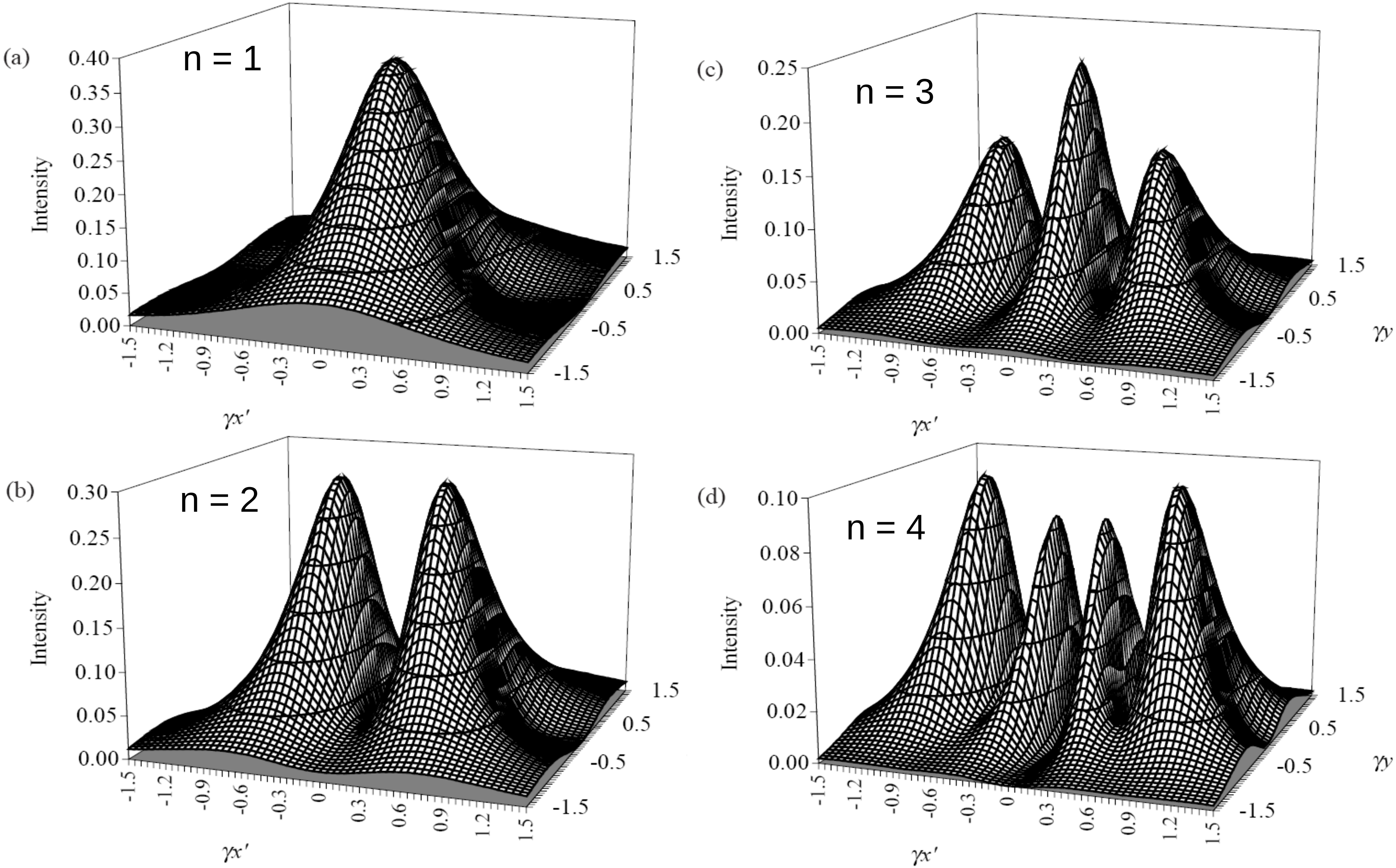
Angular Flux Density

We want to gain an appreciation for the emitted radiation from an undulator

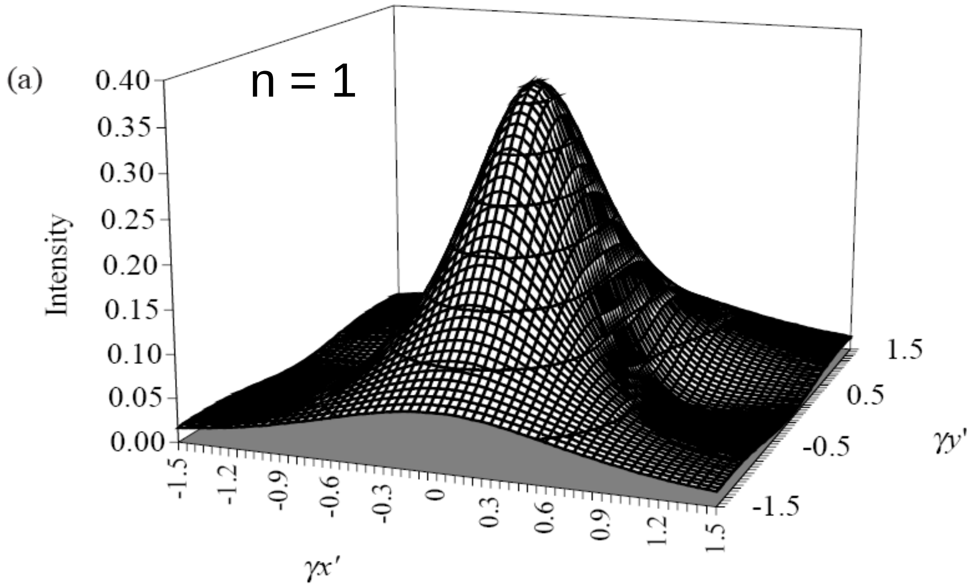
One useful parameter is the (angular) flux density (photons per solid angle) as a function of observation angle

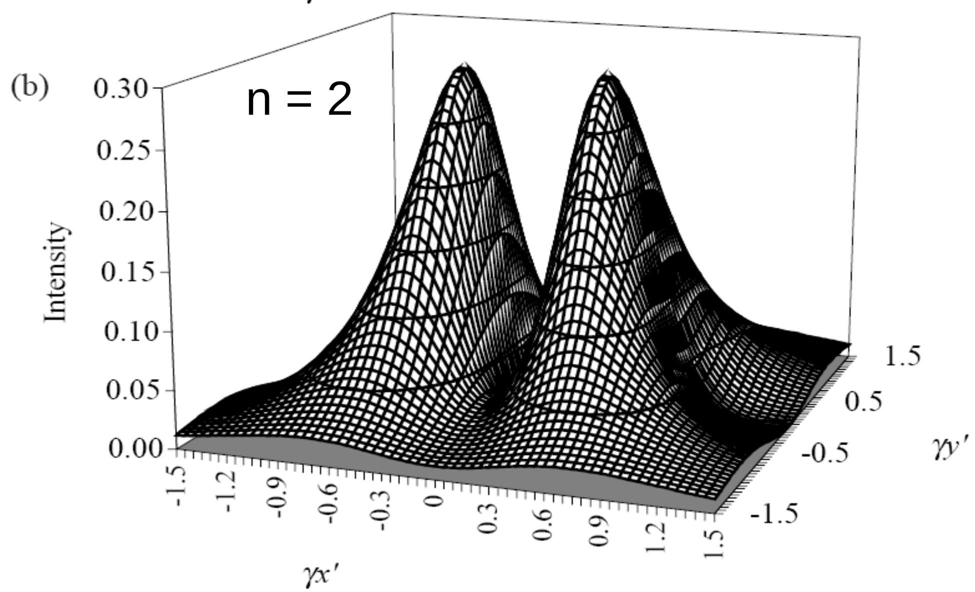
Later we will look at the flux levels and also the polarisation of the radiation

Angular Flux Density: K = 1 examples



Angular Flux Density: K = 1 examples





There are n peaks in the horizontal plane

The even harmonics have zero intensity on axis

Remember that the wavelength changes with angle (θ^2 term in the undulator equation) so these plots are **not at a fixed wavelength**

On Axis Angular Flux Density

In units of photons/sec/mrad^2/0.1% bandwidth

$$\left.\frac{d\dot{N}}{d\Omega}\right|_{\theta=0} = 1.74 \times 10^{14}\, N^2 E^2 I_b F_n(K)$$

Where:

N is the number of periods

E is the electron energy in GeV

I_b is the beam current in A

$F_n(K)$ is defined below (J are Bessel functions)

$$F_n(K) = \frac{n^2K^2}{(1+K^2/2)^2}\left(J_{(n+1)/2}(Y) - J_{(n-1)/2}(Y)\right)^2$$

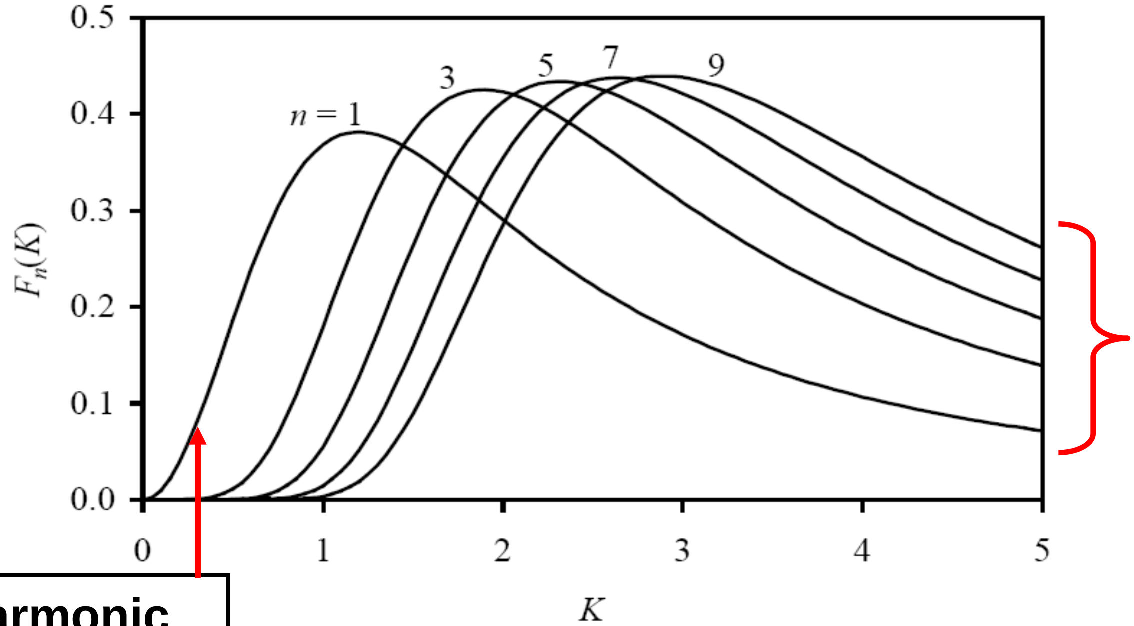
$$Y = \frac{nK^2}{4(1+K^2/2)}$$

On Axis Angular Flux Density

In units of photons/sec/mrad^2/0.1% bandwidth

$$\left.\frac{d\dot{N}}{d\Omega}\right|_{\theta=0} = 1.74 \times 10^{14}\, N^2 E^2 I_b F_n(K)$$

As K increases we can see that the influence of the higher harmonics grows



Higher harmonics have higher flux density

Only 1 harmonic at low K

Example On Axis Angular Flux Density

An Undulator with 50mm period and 100 periods with a 3GeV, 300mA electron beam will generate:

Angular flux density of **8×10^{17}** photons/sec/mrad^2/0.1% bw

For a bending magnet with the same electron beam we get a value of **~ 5×10^{13}** photons/sec/mrad^2/0.1% bw

The undulator has a flux density ~10,000 times greater than a bending magnet due to the N^2 term

Summary

The power levels emitted can be quite extreme in terms of the total power and also the power density

Insertion Devices are added to accelerators to produce light that is specifically tailored to the experimental requirements (wavelength, flux, brightness, polarisation, …)

Wavelength shifters are used to 'shift' the spectrum so that shorter wavelengths are generated

Multipole wigglers are periodic, high field devices, used to generate enhanced flux levels (proportional to the number of poles)

Undulators are periodic, low(er) field, devices which generate radiation at specific harmonics

The distinction between undulators and multipole wigglers is not black and white.