New tools for field systems in confined geometries

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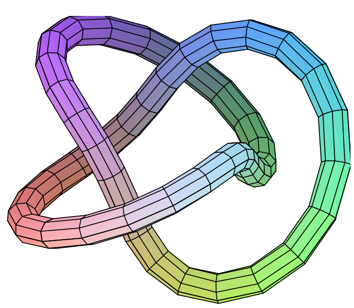
Motivation

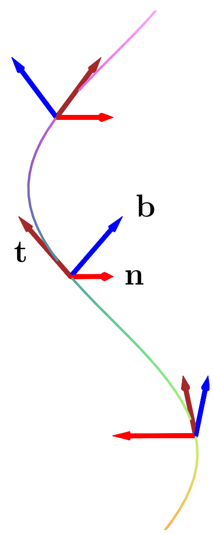
- Numerous problems in physics involve field systems in tubular regions, e.g.
  - Beam oscillations in tubular regions,
  - Spin resonances on charged orbits,
  - EM fields in wavetubes.

- Use fully covariant methods and exploit coordinate freedom.
  - Adapt coordinates to the spacetime geometry of the problem.



Coordinate systems adapted to curves

- Frenet frame $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ of a curve $\mathbf{C}(s)$:



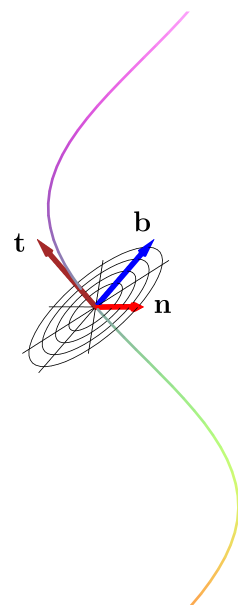
  - Frenet-Serret equations (s is arclength):

$$\mathbf{t} = \frac{d\mathbf{C}}{ds},$$
$$\frac{d}{ds}\begin{pmatrix}\mathbf{t}\\ \mathbf{n}\\ \mathbf{b}\end{pmatrix} = \begin{pmatrix}0 & \kappa & 0\\ -\kappa & 0 & \tau\\ 0 & -\tau & 0\end{pmatrix}\begin{pmatrix}\mathbf{t}\\ \mathbf{n}\\ \mathbf{b}\end{pmatrix}.$$

  - Geometry of $\mathbf{C}$ is completely specified by $\{\kappa(s), \tau(s)\}$.

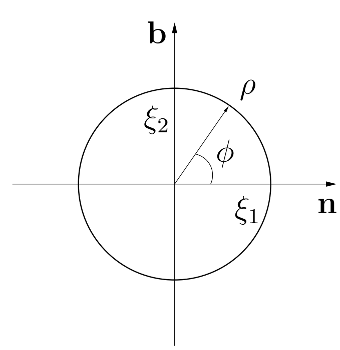
Coordinate systems adapted to curves

- Use $\{\mathbf{n}, \mathbf{b}\}$ to develop coordinate systems based on $\mathbf{C}$:



  - Frenet cartesians $\{s, \xi_1, \xi_2\}$:

$$\mathbf{r} = \mathbf{C}(\mathbf{s}) + \xi_1 \mathbf{n}(s) + \xi_2 \mathbf{b}(s).$$



Exterior differential calculus

- Differential forms are totally antisymmetric tensors and correspond directly with geometric structures.

- On $\mathbf{R}^3$:

  – 0-forms correspond to points,

  – 1-forms correspond to lines,

  – 2-forms correspond to planes,

  – 3-forms correspond to volumes,

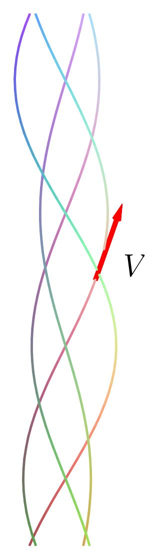
  c.f. finite element modelling.

- There exist only 3 basic operations on differential forms - d, $\star$ and $\int$:

$$dd = 0,$$
$$\star\star = \pm 1,$$
$$\int_c d\alpha = \int_{\partial c} \alpha.$$

Frames of reference on spacetime

- Constitutive properties of a medium are specified using a frame of reference on spacetime.
  - Timelike future-pointing normalized vector field V:



- V can be any frame of reference.
- Accommodates accelerating media.

Summary

- Formulate field system in covariant form on spacetime.

- Adapt the coordinate system to the geometry of the problem.

- Exploit the computational power of exterior differential calculus.

  - Only 3 operators - $d, \star$ and $\int$:

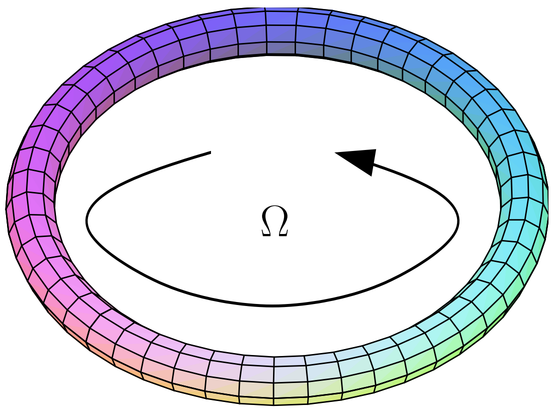
$$dd = 0,$$

$$\star\star = \pm 1,$$

$$\int_c d\alpha = \int_{\partial c} \alpha.$$

Example : Active Sagnac interferometry

- Rotating ring laser (e.g. inertial guidance)



  - Rotation splits co- and counter-rotating frequencies.

  - Classical analyses are based on ray-optics and lead to the classical Sagnac beat frequency

$$\delta\nu_{\text{Sagnac}} \simeq \frac{4\boldsymbol{\Omega}\cdot\boldsymbol{A}}{\lambda\boldsymbol{P}}. \tag{1}$$

Does a more careful field analysis yield the same results?

Large ring lasers

- UG1 at the University of Canterbury, New Zealand:

  - He-Ne ring laser ($474THz$) with dimensions $21m \times 17.5m$. Largest ring laser in the world.
  - The instantaneous direction of the Earth's rotation axis has been measured to a precision of 1 part in 10^8.
  - Lunar perturbations of the Earth's rotation and the Oppolzer modes have been measured.

Covariant Maxwell equations

- Maxwell's equations on spacetime:

$$
\begin{aligned}
dF &= 0, \\
d \star G &= j, \\
F &= -\tilde{V} \wedge E + B, \\
G &= -\tilde{V} \wedge D + H
\end{aligned}
$$

  where for a homogenous, isotropic and dispersionless medium

$$
\begin{aligned}
D &= \varepsilon_r \varepsilon_0 E, \\
B &= \mu_r \mu_0 H.
\end{aligned}
$$

- Use adapted coordinates to solve the covariant Maxwell equations.

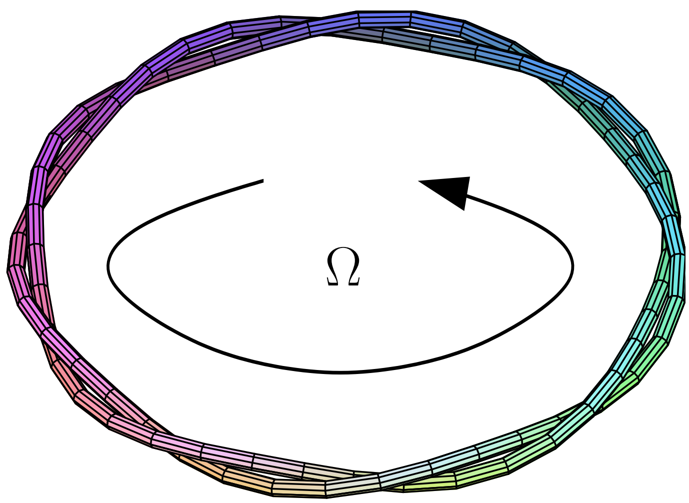
  – Obtain all the modes inside a non-planar laser cavity.

Rotating ring lasers on flat space-time

- Frequency difference for a pair of counter-propagating modes:

$$\delta\nu_{\text{Sagnac}} \simeq \frac{4\mathbf{\Omega}\cdot\mathbf{A}}{\lambda P} + \frac{cn\bar{\tau}}{\pi\mathcal{N}}.$$

- Example : crown-shaped He-Ne ring laser



  - Ring radius is 150m, fixed on the Earth and centred at the poles:

$$\delta\nu_{\text{Sagnac}} \simeq 21.0kHz + 29.6kHz.$$

Conclusion

- A general method for analysing field systems in confined geometries has been presented.
  - Covariant formulation in terms of differential forms.
  - Adapt coordinate system to geometry.
- A specific application (ring lasers) has been discussed.
  - Non-planarity of the ring laser geometry can lead to significant modifications of the classical Sagnac beat frequency.
- The tools can be applied to analyse charged beam dynamics in particle accelerators.

References

- “Twisted Electromagnetic Modes and Sagnac Ring Lasers.” DA Burton, A Noble, RW Tucker, DL Wiltshire (to submit).

- “Spin driven motion in intense spacetime wave geometries.” DA Burton, RW Tucker, C Wang. Theoretical and Applied Mechanics, 29, 77-92, 2002.