New mathematical modelling and ultra-relativistic charge

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Charged Continua with Self-Fields

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- Employs intrinsic tensor analysis and exploits the symmetries and light-cone structure of spacetime

- Offers a powerful method for the analysis of coherent radiation in complex devices controlling charged particles with laser beams

Conserved quantities

- Consider a domain of spacetime U with boundary

$$\partial U = \Sigma_1 + \Sigma_2 + \Pi$$

  for spacelike hypersurfaces Σ_i

- For $\mathcal{J}$ a closed 3-form (i.e. $d\mathcal{J} = 0$)

$$\int_{\partial U} \mathcal{J} = \int_U d\,\mathcal{J} = 0$$

- Thus

$$\int_{\Sigma_1} \mathcal{J} = \int_{-\Sigma_2} \mathcal{J} - \int_{\Pi} \mathcal{J}$$

Electromagnetic drive 3-forms

- The Maxwell field system on spacetime is $d\,F = 0$ and $d\star F = j$
- For any vector field V on spacetime and any Maxwell solution F define the "drive" 3-form

$$\tau_V = \frac{1}{2}\{i_V F \wedge \star F - i_V \star F \wedge F\}$$

- If V is a (conformal) Killing vector then

$$d\,\tau_V = -i_V F \wedge j$$

- For each (conformal) Killing vector field these equations describe a “local conservation equation”($d\tau_V = 0$) in a source free region $(j = 0)$ of spacetime

Currents and timelike Killing vectors

- If V is a timelike Killing vector can write uniquely

$$F = \tilde{E} \wedge \tilde{V} + B$$

$$\tau_V = -\tilde{E} \wedge \tilde{B} \wedge \tilde{V} + \frac{1}{2}\{g(E,E) + g(B,B)\} i_V(\star 1)$$

- $\tilde{E} \wedge \tilde{B}$ was identified by Poynting in a source free region as the local field energy transmitted normally across unit area per second (field energy current) and $\frac{1}{2}\{g(E,E) + g(B,B)\}$ the local field energy density
- More precisely $\int_\Sigma \tau_V$ is the field energy associated with the spacelike 3-chain Σ
- $\int_{S^2} i_V \tau_V$ is the power flux across an oriented spacelike 2-chain S^2

Currents and spacelike Killing vectors

- If X is a spacelike Killing vector generating spacelike translations along open integral curves then with the split:

$$\tau_X = \mu_X \wedge \tilde{V} + \mathcal{G}_X$$

- The Maxwell stress 2-form μ_X may be used to identify mechanical forces produced by a flow of field momentum with density 3-form $\mathcal{G}_X$
- In any local frame $\{X_a\}$ with dual coframe $\{e^b\}$ the 16 functions $T_{ab} = i_{X_b} \star \tau_{X_a}$ may be used to construct the second-rank stress-energy tensor

$$T = T_{ab}\, e^a \otimes e^b$$

Currents between Light-Cones

- Suppose the total system (electromagnetic field and charge) has a stress-energy tensor giving rise to a *regular* Killing current τ_K such that

  $$d\,\tau_K = 0$$

  in a domain $\mathcal{U}$ bounded by the light-like 3-chains $\Sigma(u = u_0 + \Delta u)$, $\Sigma(u = u_0)$ and the time-like 3-chain $\Sigma(r = r_0)$ for arbitrary positive constants $u_0, \Delta u, r_0$. Then

  $$\int_{\Sigma(u=u_0+\Delta u)} \tau_K - \int_{\Sigma(u=u_0)} \tau_K + \int_{\Sigma(r=r_0)} \tau_K = 0.$$

Killing Rates

- In the limit $\Delta u \longrightarrow 0$ this may be written

$$\dot{P}_K^{(\mathrm{B})}(u_0, r_0)\, d\, u + \dot{P}_K^{(\mathrm{C})}(u_0, r_0)\, d\, u = 0$$

where

$$\dot{P}_K^{(\mathrm{B})}(u_0, r_0) = \int_0^{r_0} \int_{S^2} \dot{\tau}_K(u_0, r, \theta, \phi)$$

$$\dot{P}_K^{(\mathrm{C})}(u_0, r_0) = \int_{S^2} i_{\partial_u} \tau_K(u_0, r_0, \theta, \phi)$$

Energy-Momentum Balance

- With $K \in \{\frac{\partial}{\partial y^\mu}\}$ and $\nabla \frac{\partial}{\partial y^\mu} = 0$ one has a similar balance of rates for each translational Killing vector in the basis. In terms of the “body” rate

$$\dot{P}^{(\mathrm{B})} \equiv \dot{P}^{(\mathrm{B})}_{\frac{\partial}{\partial y^\mu}}\, d\, y^\mu$$

and the “contact" rate

$$\dot{P}^{(\mathrm{C})} \equiv \dot{P}^{(\mathrm{C})}_{\frac{\partial}{\partial y^\mu}}\, d\, y^\mu$$

there is a balance of 1-forms:

$$\dot{P}^{(\mathrm{B})}(u_0, r_0) + \dot{P}^{(\mathrm{C})}(u_0, r_0) = 0.$$

Point-Particle Source

Next assume that the contact rate includes a part from the Killing currents $\tau_K^{(\mathrm{EM})}$, i.e.

$$\dot{P}^{(\mathrm{C})} = \dot{P}^{(\mathrm{C})}_{\mathrm{MECH}} + \dot{P}^{(\mathrm{C})}_{\mathrm{EM}}$$

and furthermore that this part can be calculated from the *retarded* Liénard-Wiechert solution $\mathcal{F}$ to Maxwell’s equations for an arbitrarily moving point charge q_0 in no external electromagnetic field. In adapted coordinates the solution is $\mathcal{F} = dA$ where the 1-form

$$A = \frac{q_0}{4\pi\epsilon_0} \frac{\widetilde{V}(u)}{r}.$$

Integration over the 2-chain S^2 with $r = r_0$ yields

$$\dot{P}^{(\mathrm{C})}_{\mathrm{EM}} = \frac{q_0^2}{4\pi\epsilon_0^2} \left\{ \frac{2}{3} \widetilde{\mathcal{A}}(\mathcal{A})\, \widetilde{V} + \frac{\widetilde{\mathcal{A}}}{2r_0} \right\} \Big|_{u=u_0}$$

where $\mathcal{A} = \dot{V}^\mu(u)\frac{\partial}{\partial y^\mu}$ is the acceleration field.

Regularisation

Clearly this 1-form is singular on the world-line where $r_0 = 0$. The first term however correctly accounts for the observed Larmor radiation rate of energy-momentum from an accelerating charge and is independent of r_0. One must cancel the singular rate in $\dot{P}^{\text{(C)}}_{\text{EM}}$ from singular terms in the remaining rates. One approach is to suppose that the remaining rates are determined in terms of scalar fields $\alpha(u, r)$, $\beta(u, r)$ and the vectors $V, \mathcal{A}$ such that

$$P^{\text{(B)}}(u, r_0) + P^{\text{(C)}}_{\text{MECH}}(u, r_0) = \alpha(u, r_0)\widetilde{V}(u, r_0) + \beta(u, r_0)\widetilde{\mathcal{A}}(u, r_0).$$

Lorentz-Dirac Equation

With this β and applying the projection operator $\Pi_V = 1 + \widetilde{V} \wedge i_V$ one finds

$$\beta = -\frac{2}{3}\frac{q_0^2}{4\pi\epsilon_0^2} + \frac{\dot{\alpha}}{\widetilde{\mathcal{A}}(\mathcal{A})}.$$

To cancel the exposed electromagnetic singularity take $\alpha(u, r_0) = \frac{m_0 c^2}{\epsilon_0} - \frac{q_0^2}{4\pi\epsilon_0^2}\frac{1}{2r_0}$ for some *constant* m_0 so that in the limit $r_0 \longrightarrow 0$ one has:

$$m_0 c^2 \widetilde{\mathcal{A}} = \frac{2}{3}\frac{q_0^2}{4\pi\epsilon_0}\,\Pi_V \widetilde{\dot{\mathcal{A}}}$$

or

$$m_0 c^2 \widetilde{\mathcal{A}} = -\frac{2}{3}\frac{q_0^2}{4\pi\epsilon_0}\, i_V(\widetilde{V} \wedge \widetilde{\dot{\mathcal{A}}})$$

where $\dot{\mathcal{A}} = \ddot{V}^\mu(u)\frac{\partial}{\partial y^\mu}$.

This is a system of third order differential equations for the worldline $\xi^\mu(u)$. Regarded as an initial value problem it requires unfamiliar initial data $(\xi^\mu(0), \dot{\xi}^\mu(0), \ddot{\xi}^\mu(0))$ and solutions exist corresponding to self acceleration which must be regarded as unphysical.

Lorentz-Dirac Equation

In the presence of an external Maxwell field $\mathcal{F}_{\text{ext}}$ one must confine the motion of the particle to a domain $\mathcal{U}$ that excludes the sources of $\mathcal{F}_{\text{ext}}$.

The equation of motion acquires a contribution from the Lorentz force $q_0 i_V \mathcal{F}_{\text{ext}}$:

$$m_0 c^2 \widetilde{\mathcal{A}} = q_0 i_V \mathcal{F}_{\text{ext}} + \frac{2}{3}\frac{q_0^2}{4\pi\epsilon_0}\,\Pi_V \widetilde{\dot{\mathcal{A}}}$$

so that m_0 is identified with the rest mass of the point particle.

Although solutions to this system that self-accelerate can be eliminated by demanding contrived data at different points along the world-line there remain solutions that pre-accelerate in situations where the external field is piecewise defined in spacetime.

Landau-Lifshitz Reduction

One resolution of these difficulties is to assume that the right hand side of the equation should be expanded as a series in q_0 with leading term for $\tilde{A}$ given by $\frac{q_0}{mc^2} i_V \mathcal{F}_{\text{ext}}$. Then to some order in q_0

$$\widetilde{\mathcal{A}} = \frac{q_0}{m_0 c^2} i_V \mathcal{F}_{\text{ext}} - \frac{2}{3 m_0 c^2} \frac{q_0^2}{4\pi\epsilon_0} i_V (\widetilde{V} \wedge \nabla_V \widetilde{\mathcal{A}}_{\text{ext}}) + \dots$$

where $\widetilde{\mathcal{A}}_{\text{ext}} = \frac{q_0}{m_0 c^2} i_V \mathcal{F}_{\text{ext}}$. The system is now manifestly a second order system of evolution equations. Although this offers a workable scheme it is unclear what its limitations are in different types of external field.

In situations where one has to contemplate the radiation from a large number of accelerating high-energy particles in close proximity the neglect of higher order terms in the expansion may be suspect.

Material Stress-Energy Tensor

A thermodynamically inert (cold) fluid can be modelled with the stress-energy tensor

$$T^{(\mathrm{f})} = \frac{m_0}{c\epsilon_0} \mathcal{N} \widetilde{V} \otimes \widetilde{V}$$

where $\mathcal{N}$ is a scalar number density field, m_0 some constant with the dimensions of mass, V the unit time-like 4-velocity field of the fluid and $g(V, V) = -1$. Such a stress-energy tensor gives rise to a set of Killing currents

$$\tau^{(\mathrm{f})}_{K_\mu} = \frac{m_0}{c\epsilon_0} g(V, K_\mu) \star (\mathcal{N}\widetilde{V})$$

which are added to $\tau^{(\mathrm{EM})}_{K_\mu}$ to yield the total set of Killing currents for the interacting system.

Charged Fluid Dynamics

If one assumes that the electric current 3-form is $j = q_0 \mathcal{N} \star \widetilde{V}$ for some electric charge constant q_0 and that $\mathcal{N}$ is *regular* then the conservation laws

$$d\, j = 0$$

$$d(\tau^{(\mathrm{EM})}_{K_\mu} + \tau^{(\mathrm{f})}_{K_\mu}) = 0$$

yield the field equation of motion

$$\nabla_V \widetilde{V} = \frac{q_0}{m_0 c^2} i_V \mathcal{F}.$$

This equation must be solved consistently with the Maxwell equations to determine V, $\mathcal{N}$ and $\mathcal{F}$ for prescribed initial and boundary conditions.

The Charged Fluid System

$$
\begin{aligned}
dF &= 0\,,\\
d\star F &= -\rho\star\widetilde{V}\,,\\
\nabla_V\widetilde{V} &= \ i_V F\,,\\
V\cdot V &= -1
\end{aligned}
$$

for the triple (V,ρ,F). These leads immediately to the integrability condition (conservation of electric charge)

$$d\star(\rho\widetilde{V}) = 0.$$

Field Equations for a Charged Fluid in terms of p, ρ, e and b

$$\boldsymbol{\nabla}\cdot\boldsymbol{e}=\frac{m_0c^2}{q_0}\gamma\rho,\qquad \boldsymbol{\nabla}\times\boldsymbol{b}=\frac{1}{q_0}\rho\boldsymbol{p}+\frac{1}{c^2}\frac{\partial\boldsymbol{e}}{\partial t},$$

$$\boldsymbol{\nabla}\times\boldsymbol{e}+\frac{\partial\boldsymbol{b}}{\partial t}=0,\qquad \boldsymbol{\nabla}\cdot\boldsymbol{b}=0,$$

$$\gamma\frac{\partial\boldsymbol{p}}{\partial t}+\left(\frac{\boldsymbol{p}}{m_0}\cdot\boldsymbol{\nabla}\right)\boldsymbol{p}=q_0\left(\gamma\boldsymbol{e}+\frac{1}{m_0}\boldsymbol{p}\times\boldsymbol{b}\right),$$

$$-\gamma^2+\frac{\boldsymbol{p}\cdot\boldsymbol{p}}{m_0^2c^2}=-1,$$

$$m_0\frac{\partial}{\partial t}(\gamma\rho)+\boldsymbol{\nabla}\cdot(\rho\boldsymbol{p})=0$$

Exact Symmetric Solutions

- Plane symmetric solutions reduce the system to a field theory on a 2-dimensional Lorentzian spacetime (with global coordinates t, z).
- The system is solved exactly using a co-moving coordinate system (τ, σ) adapted to the charged continuum.
- However, expressing the solutions in terms of laboratory coordinates (t, z) requires the inverse of the mapping $(\tau, \sigma) \longrightarrow (t, z)$, which is generally difficult to obtain in closed form.
- A running parameter $\varepsilon > 0$ is introduced into the mapping $(\tau, \sigma) \longrightarrow (t, z)$ and a perturbation scheme in ε facilitates an order-by-order construction of the inverse of the mapping $(\tau, \sigma) \longrightarrow (t, z)$ leading to 1-parameter families $(V^\varepsilon, \rho^\varepsilon, F^\varepsilon)$ of solutions in ε.
- $$F^\varepsilon = \Sigma_{n=-1}^{\infty} \varepsilon^n F_n, \quad V^\varepsilon = \Sigma_{n=-1}^{\infty} \varepsilon^n V_n, \quad \rho^\varepsilon = \Sigma_{n=1}^{\infty} \varepsilon^n \rho_n$$

  over some range of ε where the coefficients F_n, V_n and ρ_n are 2-forms, vector fields and scalar fields respectively.

Exact Solutions

Exact solutions take the form

$$F = \mathcal{E}(t, z)\, dt \wedge dz,$$

$$V = \frac{1}{\sqrt{1 - \mu^2(t, z)}} \left(\partial_t + \mu(t, z)\partial_z\right)$$

where μ is the magnitude of the Newtonian velocity field of the charged continuum measured by an inertial (laboratory) observer.

The electric field satisfies

$$d\mathcal{E} = \rho \# \widetilde{V},$$

$$\nabla_V \widetilde{V} = \mathcal{E} \# \widetilde{V}$$

where $\#$ is the Hodge map associated with the volume 2-form $\#1 \equiv dt \wedge dz$ and $\mathcal{E}$ is constant along the integral curves of V.

Exact Solutions

Particular solutions satisfying prescribed initial conditions yield *implicit* relations between co-moving and inertial descriptions:

$$t = \hat{t}(\tau, \sigma) = \frac{1}{\zeta(\sigma)} \sinh(\zeta(\sigma)\tau)\,,$$

$$z = \hat{z}(\tau, \sigma) = \frac{1}{\zeta(\sigma)} \left[\cosh(\zeta(\sigma)\tau) - 1\right] + \sigma$$

where the function $\zeta : \mathbb{R} \to \mathbb{R}$ must be supplied as initial data. After ζ has been specified, expressions for V and F in the (t, z) coordinate system are obtained by inverting the above to give (τ, σ) in terms of (t, z):

$$\tau = \hat{\tau}(t, z),$$

$$\sigma = \hat{\sigma}(t, z)$$

and then

$$\mathcal{E}(t, z) = \zeta\left(\hat{\sigma}(t, z)\right).$$

Asymptotic Expansions

Since the electric field $\mathcal{E}^\varepsilon(t,z)$ and 4-velocity field V^ε can be written entirely in terms of $\hat{\sigma}^\varepsilon$, t and z:

$$\mathcal{E}^\varepsilon(t,z) = \zeta^\varepsilon\left(\hat{\sigma}^\varepsilon(t,z)\right)$$

$$V^\varepsilon = \sqrt{1+[\zeta^\varepsilon\left(\hat{\sigma}^\varepsilon(t,z)\right)]^2t^2}\partial_t + \zeta^\varepsilon\left(\hat{\sigma}^\varepsilon(t,z)\right)t\partial_z$$

the inertial solution is obtained order-by-order in ε using

$$\hat{\sigma}^\varepsilon(t,z) = \Sigma_{n=0}^{\infty}\varepsilon^n\hat{\sigma}_n(t,z)$$

Asymptotic Solutions

Equating orders in ε:

$$\hat{\sigma}^{\varepsilon}(t,z) = z - t + \frac{\varepsilon}{\zeta_{-1}} - \frac{1 + 2t\zeta_0(z-t)}{2t\zeta_{-1}^2}\varepsilon^2 + O(\varepsilon^3)$$

$$\mathcal{E}^{\varepsilon}(t,z) = \frac{1}{\varepsilon}\zeta_{-1} + \zeta_0(z-t) + \frac{\zeta_0'(z-t)}{\zeta_{-1}}\varepsilon + O(\varepsilon^2)$$

$$V^{\varepsilon} = \left(\frac{1}{\varepsilon}\zeta_{-1} + \zeta_0(z-t)\right) t\left(\partial_t + \partial_z\right)$$

$$+ \left(\frac{1 + 2t^2\zeta_0'(z-t)}{2t\zeta_{-1}}\partial_t + \frac{\zeta_0'(z-t)t}{\zeta_{-1}}\partial_z\right)\varepsilon + O(\varepsilon^2)$$

where $\zeta_0'(z) \equiv \frac{d\zeta_0}{dz}(z)$.

Asymptotic Solutions

The electric current $J^\varepsilon \equiv \rho^\varepsilon V^\varepsilon$ is

$$J^\varepsilon = \rho^\varepsilon V^\varepsilon = \widetilde{\#d\mathcal{E}^\varepsilon}$$

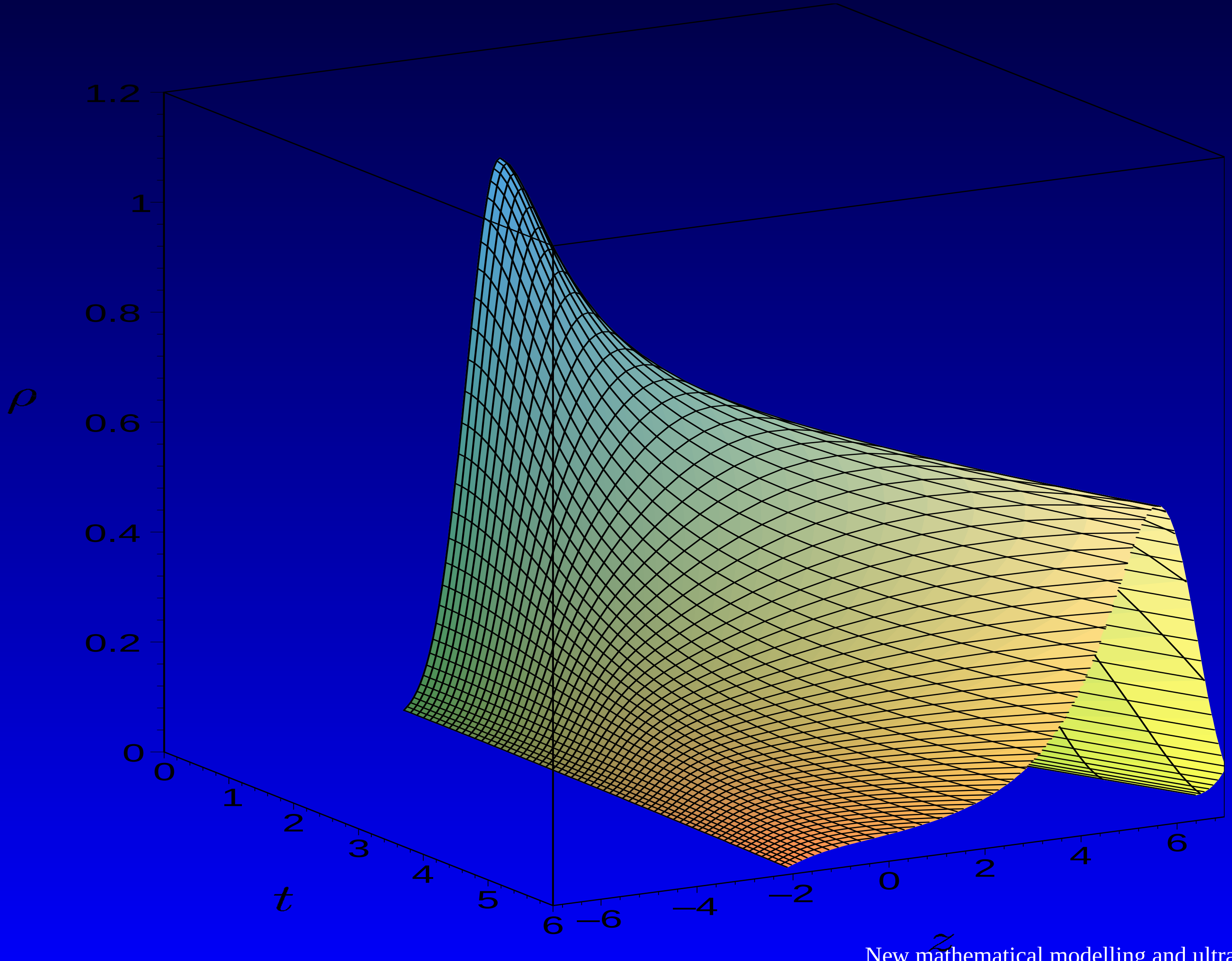
$$= \zeta_0'(z-t)\,(\partial_t + \partial_z) + \varepsilon \frac{\zeta_0''(z-t)}{\zeta_{-1}}\,(\partial_t + \partial_z) + O(\varepsilon^2).$$

and

$$\rho^\varepsilon = \varepsilon \frac{\zeta_0'}{\zeta_{-1} t} + O(\varepsilon^2).$$

Charged Bunch

Charge Density ρ moving Under Self and External Fields



Proposed New Representation

This example suggests that more general solutions be represented order-by-order in ε with.

$$F^\varepsilon = \Sigma_{n=-1}^\infty \varepsilon^n F_n, \qquad V^\varepsilon = \Sigma_{n=-1}^\infty \varepsilon^n V_n, \qquad \rho^\varepsilon = \Sigma_{n=1}^\infty \varepsilon^n \rho_n$$

where F_{-1} is an external field (a solution to the source-free Maxwell equations).

The electric 4-current then has the form

$$J^\varepsilon = \rho^\varepsilon V^\varepsilon = \Sigma_{n=0}^\infty \varepsilon^n J_n.$$

The above expansions partially decouple the non-linear field system yielding an infinite hierarchy of equations that are amenable to solution when supplemented with appropriate boundary conditions and initial data.

Perturbation about the Light Cone

- There exists a class of solutions representing configurations of charged particles in *ultra-relativistic* collective motion.
- An ultra-relativistic velocity field is a pointwise limiting concept that depends on the existence of the forward light-cone structure at each event in spacetime.
- To leading order in the expansion defined below the velocity field of the charged continuum is light-like.
- The full series will be considered as an asymptotic expansion for a solution and physically represents an ultra-relativistic configuration controlled by the parameter ε.
- Such configurations are chosen to be representative of the class relevant to charged beams in high-energy accelerators.

Perturbation about the Light Cone

Introduce perturbation series for $(V^\varepsilon, \rho^\varepsilon, F^\varepsilon)$ in ε of the form

$$V^\varepsilon = \sum_{n=-1}^{\infty} \varepsilon^n V_n$$

$$\rho^\varepsilon = \sum_{n=1}^{\infty} \varepsilon^n \rho_n$$

$$F^\varepsilon = \sum_{n=-1}^{\infty} \varepsilon^n F_n$$

The Hierarchy

The coefficients in the expansions satisfy

$$dF_{n-1} = 0 \qquad \text{for } n \in \{0,1,2,\ldots\}$$

$$d\star F_{n-1} = \begin{cases} 0 & \text{for } n = 0 \\ -\Sigma_{r=1}^{n} \star \rho_r \widetilde{V}_{n-r-1} & \text{for } n \in \{1,2,\ldots\} \end{cases}$$

$$\Sigma_{r=0}^{n} \nabla_{V_{r-1}} \widetilde{V}_{n-r-1} = \Sigma_{r=0}^{n} i_{V_{r-1}} F_{n-r-1} \qquad \text{for } n \in \{0,1,2,\ldots\}$$

$$\Sigma_{r=0}^{n} V_{r-1} \cdot V_{n-r-1} = \begin{cases} -1 & \text{for } n = 2 \\ 0 & \text{for } n \in \{0,1,3,4,5,\ldots\} \end{cases}$$

$$\Sigma_{r=1}^{n+1} d\star \left(\rho_r \widetilde{V}_{n-r}\right) = 0 \qquad \text{for } n \in \{0,1,2,\ldots\}$$

These partially decoupled equations form a hierarchy and are amenable to an ordered analysis.

Example

Solutions, describing an ultra-relativistic charged distribution propagating along the z-axis with its electromagnetic self-fields have:

$$\begin{aligned} V^\varepsilon &= \frac{1}{\varepsilon} 2\gamma_{-1}\partial_v - \varepsilon \frac{1}{2\gamma_{-1}}\partial_u + O(\varepsilon^2) \\ &= \left(\frac{1}{\varepsilon}\gamma_{-1} + \frac{\varepsilon}{4\gamma_{-1}}\right)\partial_t + \left(\frac{1}{\varepsilon}\gamma_{-1} - \frac{\varepsilon}{4\gamma_{-1}}\right)\partial_z + O(\varepsilon^2) \\ F^\varepsilon &= d\Phi_0 \wedge du + \varepsilon d\Phi_1 \wedge du + O(\varepsilon^2) \\ &= -(d\Phi_0 + \varepsilon d\Phi_1) \wedge dt + (d\Phi_0 + \varepsilon d\Phi_1) \wedge dz + O(\varepsilon^2) \end{aligned}$$

where Φ_0 and Φ_1 satisfy

$$d_\perp \#_\perp d_\perp \Phi_0 = \gamma_{-1}\rho_1 \#_\perp 1, \quad d_\perp \#_\perp d_\perp \Phi_1 = \gamma_{-1}\rho_2 \#_\perp 1$$

and

$$\begin{aligned} &\Phi_0 = \hat{\Phi}_0(u,x,y), \quad \Phi_1 = \hat{\Phi}_1(u,x,y), \\ &\rho_1 = \hat{\rho}_1(u,x,y), \quad \rho_2 = \hat{\rho}_2(u,x,y), \\ &\gamma_{-1} = \hat{\gamma}_{-1}(x,y) \qquad u \equiv z - t \end{aligned}$$

Example

- The scalar fields $\rho_1, \rho_2, \gamma_{-1}$ are determined by their values on the space-like hypersurface $t = 0$ given as data.
- The potentials Φ_0 and Φ_1 are then solved in terms of $\gamma_{-1}\rho_1$ and $\gamma_{-1}\rho_2$ using 2-dimensional Poisson equations in the (x, y) plane.
- The 3-velocity of the beam is along the direction z in the laboratory frame with Newtonian speed

$$\frac{\frac{1}{\varepsilon}\gamma_{-1} - \varepsilon\frac{1}{4\gamma_{-1}} + O(\varepsilon^2)}{\frac{1}{\varepsilon}\gamma_{-1} + \varepsilon\frac{1}{4\gamma_{-1}} + O(\varepsilon^2)} = 1 - \frac{\varepsilon^2}{2\gamma_{-1}^2} + O(\varepsilon^3)$$

Compact Charged Bunch

- Take a Gaussian bunch with transverse radius R_0 travelling at constant Newtonian speed $1-\frac{\varepsilon^2}{2b_0^2}$ to order ε^2:

  $$\hat{\gamma}_{-1}(x,y)=b_0, \quad \hat{\rho}_1(z,x,y)=a_0\exp\left(-\frac{x^2+y^2}{R_0^2}\right)\Xi(z)$$

  where a_0, R_0 and b_0 are constants and $\Xi:\mathbb{R}\to\mathbb{R}$ is a smooth bump function vanishing outside the interval $(-z_1,z_1)$ and $\Xi(z)=1$ for $z\in(-z_2,z_2)$ and $z_1>z_2>0$.

- Then the laboratory reduced charge density $\gamma_{-1}\rho_1$ for some range of t is

  $$\gamma_{-1}\rho_1=\hat{\gamma}_{-1}(x,y)\hat{\rho}_1(z-t,x,y)=a_0\,b_0\exp\left(-\frac{x^2+y^2}{R^2}\right)\Xi(z-t).$$

Example

- In cylindrical polar coordinates (t, R, ϕ, z) with $x = R\cos\phi$ and $y = R\sin\phi$, a cylindrically symmetric solution, well-behaved at $R = 0$, is

$$\Phi_0 = \left\{ \int_0^R a_0\, b_0\, \frac{R_0^2}{2s} \left[1 - \exp\left(-\frac{s^2}{R_0^2} \right) \right] ds \right\} \Xi(z - t)$$

- The corresponding electromagnetic 2-form F_0 is

$$F_0 = a_0\, b_0\, \frac{R_0^2}{2R} \left[1 - \exp\left(-\frac{R^2}{R_0^2} \right) \right] \Xi(z - t)\, dR \wedge (-dt + dz).$$

- The laboratory electric field is radial, the magnetic field is azimuthal and their magnitudes are equal and vanish outside of the support of Ξ.

Conclusions

- A description based on charged continua rather than classical point particles has been explored.
- Pathologies (such as pre-acceleration) associated with radiating point particles are avoided by relying on field-theoretical notions.
- A novel analysis of charged beam dynamics has been presented and a model of a freely propagating charged bunch discussed. The approach relies on an asymptotic series representation of solutions to self-consistent spacetime covariant field equations for a charged continuum.

Conclusions

- The asymptotic series for V is based on the light-like vector field V_{-1} leading to an ultra-relativistic approximation.
- The hierarchy of equations obtained are more amenable to analysis than the original non-linear field system and particular solutions have been presented.
- Avenues for development include ultra-relativistic charged beams in the vicinity of beam pipes, RF cavities, spoilers, etc. leading to dynamical effects that are often described in terms of “wake-fields” and a clearer understanding of radiation-reaction exhibited by continuum models of charged particle beams.

Reference

Asymptotic analysis of ultra-relativistic charge

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