# Pondermotive Forces in Moving Media and Gravitational Interactions

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# Historical Controversy

- What is the electromagnetic momentum density in a polarizable and magnetizable medium?
- $(\mathbf{D} \times \mathbf{B})$ : Minkowski (1908)
- $(\mathbf{E} \times \mathbf{H})/\mathbf{c^2}$: Abraham (1908)
- $(\mathbf{D} \times \mathbf{B})$: Laue and Moller (1952)
- $(\mathbf{D} \times \mathbf{B})$: Pauli (1958)
- $(\mathbf{E} \times \mathbf{H})/\mathbf{c^2}$: Landau and Lifshitz (1960)
- $\epsilon_0(\mathbf{E} \times \mathbf{B})$: D Nelson (1991)

# Experimental Investigations

- What is the electromagnetic "radiation pressure" on a polarisable and magnetisable medium?
- Jones and Richards (1954): Observed light pressure on objects immersed in liquids. Claimed results in agreement with Minkowski.
- Ashkin and Dzieddzic (1973): Measured radiation pressure on free liquid surface. Claimed results in agreement with Minkowski
- Haus (1969 ) and Gordon (1973) accept Abraham but argue that material mechanical momentum at contacts contribute to light forces in liquids.
- Burt and Peierls (1973) accept Abraham but reject measurable contributions from liquid motions in the Jones and Richards experiment and leave the discrepancy between Abrahams's theory unresolved.

# The Electro-magnetic "drive forms" in vacuo

- The basic properties of the electromagnetic stress-energy-momentum tensor can be succinctly discussed in terms of a set of electromagnetic stress-energy-momentum $3-$forms. In vacuo the Maxwell field system with a $3-$form electric current source $j$ satisfies

$$d\,F = 0$$

  and

$$d \star F = j.$$

- For any vector field $V$ on spacetime and any Maxwell solution $F$ to this system one can introduce a “drive" 3-form

$$\tau_V^{EM} = \frac{1}{2}\{i_V F \wedge \star F - i_V \star F \wedge F\}.$$

  $\star$ denotes the Hodge map associated with the spacetime metric $g$.

# Conserved Flows and Electric and Magnetic Fields in vacuo

- If $K$ is any (conformal) Killing vector on a domain of spacetime it then follows simply that

$$d\,\tau_K^{EM} = -i_K\,F \wedge j$$

- For each (conformal) Killing vector field these equations describe a “local conservation equation”( $d\tau_K = 0$) in a source free region ($j = 0$).
- The electric **e** and magnetic **b** 1-forms on spacetime are defined in terms of the Maxwell 2-form $F$ and an arbitrary *unit* timelike (observer) $4-$ velocity vector field $U$ by

$$F = \mathbf{e} \wedge \tilde{U} - \star(\mathbf{b} \wedge \tilde{U}) \tag{1}$$

where $\tilde{U} = g(-, U)$, $i_U\mathbf{e} = 0, i_U\mathbf{b} = 0$.

# Energy Density and Power flow in vacuo

- For $V$ any *timelike* Killing vector one has

$$\tau_V^{EM} = -\mathbf{e} \wedge \mathbf{h} \wedge \tilde{V} + \frac{1}{2}\{g(\tilde{\mathbf{e}}, \tilde{\mathbf{e}}) + g(\tilde{\mathbf{b}}, \tilde{\mathbf{b}})\} i_V(\star 1)$$

- The $2-$ form $\mathbf{e} \wedge \mathbf{h}$ was identified by Poynting in a source-free region as the local field energy transmitted normally across unit area per second (field energy current) and $\frac{1}{2}\{g(\tilde{\mathbf{e}}, \tilde{\mathbf{e}}) + g(\tilde{\mathbf{b}}, \tilde{\mathbf{b}})\}$ the local field energy density.
- More precisely $\int_\Sigma \tau_V$ is the field energy associated with the spacelike 3-chain $\Sigma$ and $\int_{S^2} i_V \tau_V$ is the power flux across an oriented spacelike 2-chain $S^2$.

# Forces and Momentum Flow in vacuo

- If $X$ is a *spacelike* translational Killing vector generating spacelike translations along open integral curves then with the split:

  $$\tau_X^{EM} = \mu_X \wedge \tilde{V} + \mathcal{G}_X$$

  the Maxwell stress 2-form $\mu_X$ may be used to identify mechanical forces produced by a flow of field momentum with density 3-form $\mathcal{G}_X$.
- N.B. The forms $\mu_X$ and $\mathcal{G}_X$ are ab-initio unrelated.

# The Electromagnetic Stress-energy-momentum Tensor in vacuo

- In general, in the absence of Killing vectors one looses strictly conserved currents (closed $3-$ forms) but a set of four local $3-$ form currents $\tau^{EM}_{X_c}$ can be defined in any local coframe.
- In any frame $\{X_a\}$ with dual coframe $\{e^b\}$ the 16 functions $T^{EM}_{ab} = i_{X_b} \star \tau^{EM}_{X_a}$ may be used to construct the second-rank stress-energy-momentum tensor

$$T^{EM} = T^{EM}_{ab}\, e^a \otimes e^b.$$

- It is symmetric if $\tau_{X_a} \wedge e_b = \tau_{X_b} \wedge e_a$.

# Matter, Electromagnetic Fields and Pondermotive Forces

- If a coupled system of electromagnetic , gravitational and matter fields has a total stress-energy-momentum tensor

$$T^{Total} = T^{EM} + T^{m+int}$$

then on general grounds one has:

$$\nabla \cdot T^{Total} = 0 \qquad (2)$$

in terms of a (Koszcul) connection $\nabla$ on spacetime.

- Different authors partition the total stress-energy-momentum tensor into a sum of partial stress-energy-momentum tensors in different ways. Such partial stress-energy-momentum tensors need not be symmetric.
- The divergences of certain partial stress-energy-momentum tensors are sometimes called *Pondermotive* forces.

# The Abraham and Minkowski Tensors in Media

- Abraham introduced the symmetric EM stress-energy-momentum for a medium with 4-velocity $V$:

$$T^{em} = i_a F \otimes i^a G + i_a G \otimes i^a F + \star(F \wedge \star G) g + \widetilde{V} \otimes \mathbf{s} + \mathbf{s} \otimes \widetilde{V} \quad (3)$$

where

$$\mathbf{s} = \star(\mathbf{e} \wedge \mathbf{h} \wedge \widetilde{V}) + \star(\mathbf{b} \wedge \mathbf{d} \wedge \widetilde{V}) \quad (4)$$

and

$$\mathbf{d} = i_V G\,, \qquad \text{and} \qquad \mathbf{h} = i_V \star G\,. \quad (5)$$

so that

$$G = \mathbf{d} \wedge \tilde{V} - \star(\mathbf{h} \wedge \tilde{V}) \quad (6)$$

- By contrast Minkowski introduced the stress-energy-momentum tensor

$$T^{em} = i_a G \otimes i^a F + \star(F \wedge \star G) g \quad (7)$$

which is not symmetric.

# The Relevance of Einstein Gravity

- If the divergence operator $(\nabla \cdot)$ is induced from a connection on the bundle of linear frames over spacetime and is metric compatible and torsion free with gravitational fields satisfying Einstein's equations

$$Ein = T^{Total}$$

  then $T^{Total}$ is a symmetric stress-energy-momentum tensor $T^{Total}_{ab} = T^{Total}_{ba}$.
- Any such symmetric tensor can be partitioned into non-symmetric partial tensors in infinitely many ways.
- Such a partition is then an expediency without fundamental significance.

# The Relevance of non-Einstein Gravity

- In theories of gravitation based on non-pseudo-Riemannian geometries the *natural connection* may have torsion.
- For example in an Einstein-Cartan theory with matter (e.g. spinor fields) that gives rise to a connection with torsion, the generalised Einstein tensor $Ein^{EC}$ determined by varying the generalised Einstein-Hilbert action with respect to ortho-normal coframes is non-symmetric and hence the source tensor $T^{EC}$ defined by

$$Ein^{EC} = T^{EC}$$

is similarly non-symmetric.

# The Definition of the Total Stress-energy-momentum Tensor

- For some forms of gravitational-matter couplings the variation of the total action with respect to the connection gives rise to algebraic equations for the connection.
- In principle these can be solved for the connection which can always be decomposed into a sum containing the torsion-free metric-compatible (Levi-Civita) connection used in Einstein's pseudo-Riemannian description of gravitation.
- The generalised Einstein tensor $Ein^{EC}$ can then be written $Ein + S$ in terms of the Einstein tensor $Ein$ and the above becomes

$$Ein = T^E \tag{8}$$

  where $T^E \equiv T^{EC} - S$ is symmetric and divergenceless with respect to the Levi-Civita connection.
- In such cases one may define the total stress-energy-momentum tensor as the source tensor $T^E$ for Einstein's equations. It is then by definition symmetric.

# The Definition of the Total Stress-energy-momentum Tensor

- If the natural connection (determined by a connection variation of the total action) gives rise to dynamic torsion, determined by a partial differential system involving all fields, the reduction to a geometrical formulation in terms of a metric and Levi-Civita connection becomes an impracticality.
- In such a situation the definition of the total stress-energy-momentum tensor is best left as $T^{EC}$.
- This is divergence-less with respect to $\nabla$ but not in general symmetric.

# Action Principles

- For a polarizable and magnetizable medium constructing a total action that incorporates electromagnetic interactions with all material fundamental constituents is not feasible
- Standard methods construct phenomenological partial stress-energy-momentum tensors based on either coarse-graining detailed interactions between fields or the introduction of effective degrees of freedom.
- Phenomenological stress-energy-momentum tensors are often of greater value that actions based on “fundamental fields” since they can often be related more directly to experiment.
- Although gravitation may be an ignorable interaction and we accommodate electromagnetic properties of a medium into permittivity and permeability tensors we demand that the action describing electromagnetic interactions with media give rise by variation to Maxwell’s covariant phenomenological equations $d\,F = 0, d \star G = j$ *and* a symmetric stress-energy-momentum tensor.

# Drive Forms from an Action 4-Form on Spacetime

- Suppose the total action for a medium contains a term $\int \Lambda^{em}$ that gives the phenomenological Maxwell equations by variation. Then we have

$$\dot{\Lambda}^{em} = \dot{e}^a \wedge \tau_a \tag{9}$$

  where dot refers to variation with respect to the coframe $\{e^0, e^1, e^2, e^3\}$.
- If $\Lambda^{em}$ involves only the (pseudo-) Riemannian connection expressed in terms of the metric and the metric variation is induced by an ortho-normal frame variation:

$$\tau_a = -2 i_{X_b} \left( \frac{\delta \Lambda}{\delta g_{bc}} \right) g_{ac} \tag{10}$$

- Since

$$\tau_a \wedge e_b = 2 \left( \frac{\delta \Lambda}{\delta g_{dc}} \right) g_{ac} g_{bd} \tag{11}$$

  it is clear that the stress-energy-momentum tensor obtained from metric variations is symmetric.

# Linear (non-dispersive) Inhomogeneous Anisotropic Media

- *Linear* constitutive relations are specified by

$$G = Z(F) \tag{12}$$

where is $Z$ is a tensor that maps $2-$forms to $2-$forms: $\{e^0, e^1, e^2, e^3\}$ via

$$\frac{1}{2} G_{ab} e^a \wedge e^b = \frac{1}{4} Z^{cd}{}_{ab} F_{cd} e^a \wedge e^b \tag{13}$$

where

$$Z^{cd}{}_{ab} = -Z^{cd}{}_{ba} = -Z^{dc}{}_{ab} = Z^{dc}{}_{ba} \tag{14}$$

i.e.

$$\begin{aligned} \mathbf{d} &= \zeta^{\mathrm{de}}(\mathbf{e}) + \zeta^{\mathrm{db}}(\mathbf{b}) \\ \mathbf{h} &= \zeta^{\mathrm{he}}(\mathbf{e}) + \zeta^{\mathrm{hb}}(\mathbf{b}) \end{aligned} \tag{15}$$

# Symmetry Conditions

- If $\zeta^{\text{db}}$ and $\zeta^{\text{he}}$ are non-zero in the co-moving frame of the medium it is called magneto-electric.
- If $\zeta^{\text{db}}$ and $\zeta^{\text{he}}$ are zero in the co-moving frame of the medium it is called non-magneto-electric.
- The spatial tensors $\zeta^{\text{db}}$ and $\zeta^{\text{he}}$ may be non-zero in a non-comoving frame for a non-magneto-electric medium
- Thermodynamic and time symmetry conditions impose the relation $Z = Z^\dagger$ or

$$\zeta^{\text{de}\,\dagger} = \zeta^{\text{de}}, \quad \zeta^{\text{hb}\,\dagger} = \zeta^{\text{hb}} \quad \text{and} \quad \zeta^{\text{db}\,\dagger} = -\zeta^{\text{he}} \tag{16}$$

in all spacetime frames where the adjoint $T^\dagger$ of a tensor $T$ which maps $p-$forms to $p-$forms, is defined by:

$$\alpha \wedge \star T(\beta) = \beta \wedge \star T^\dagger(\alpha) \qquad \text{for all } \alpha, \beta \in \Gamma\Lambda^p\mathcal{M} \tag{17}$$

# Variational Maxwell System

- With $F = d\,A$ and the constitutive relations $G = Z(F)$ with $Z = Z^{\dagger}$, consider the action

$$S[A,g] = \int_M \tfrac{1}{2} F \wedge \star G$$

- With prime denoting the variation with respect to $A$:

$$\begin{aligned}\Lambda' =& \tfrac{1}{2}\Big(dA' \wedge \star Z(dA) + dA \wedge \star Z(dA')\Big)\\ =& dA' \wedge \star Z(dA) = A' \wedge d \star Z(dA) = A' \wedge d \star G\end{aligned}$$

- Hence the source-free Maxwell system follows from this variation

$$d\,F = 0 \qquad \text{and} \qquad d \star G = 0. \tag{18}$$

# The Dependence of $\zeta_t^{\rm de}$, $\zeta_t^{\rm db}$, $\zeta_t^{\rm he}$ and $\zeta_t^{\rm hb}$ on the metric

Demand that the lifted tensors $\zeta_t^{\rm de}$, $\zeta_t^{\rm db}$, $\zeta_t^{\rm he}$ and $\zeta_t^{\rm hb}$ satisfy for all $X$ and $\alpha$:

- $\zeta_t|_{t=0} = \zeta_0 \qquad$ for $\quad \zeta_t = \zeta_t^{\rm de}, \zeta_t^{\rm db}, \zeta_t^{\rm he}, \zeta_t^{\rm hb}$
- Map the space orthogonal to $\tilde{V}_t$ to itself
  $i_{V_t}\zeta_t(\alpha) = 0 \qquad$ for $\quad \zeta_t = \zeta_t^{\rm de}, \zeta_t^{\rm db}, \zeta_t^{\rm he}, \zeta_t^{\rm hb}$
- Maintain symmetry : $(\zeta_t^{\rm hb})^{\dagger_t} = \zeta_t^{\rm hb}$ and $(\zeta_t^{\rm db})^{\dagger_t} = -\zeta_t^{\rm he}$

With the projector $\pi_t = {\rm Id} + \tilde{V}_t \otimes V_t$ these requirements are satisfied if:

$$i_X\zeta_t(\alpha) = \tfrac{1}{2}\left(i_X\zeta_0(\pi_t\alpha) + i_{g_t^{-1}\alpha}(\zeta_0)^{\dagger_0}(\pi_t g_t X)\right) \tag{19}$$

i.e.

$$\begin{aligned}
i_X\zeta_t^{\rm de}(\alpha) &= \tfrac{1}{2}\left(i_X\zeta_0^{\rm de}(\pi_t\alpha) + i_{g_t^{-1}\alpha}\zeta_0^{\rm de}(\pi_t g_t X)\right) ,\\
i_X\zeta_t^{\rm hb}(\alpha) &= \tfrac{1}{2}\left(i_X\zeta_0^{\rm hb}(\pi_t\alpha) + i_{g_t^{-1}\alpha}\zeta_0^{\rm hb}(\pi_t g_t X)\right) ,\\
i_X\zeta_t^{\rm db}(\alpha) &= \tfrac{1}{2}\left(i_X\zeta_0^{\rm db}(\pi_t\alpha) - i_{g_t^{-1}\alpha}\zeta_0^{\rm he}(\pi_t g_t X)\right) ,\\
i_X\zeta_t^{\rm he}(\alpha) &= \tfrac{1}{2}\left(i_X\zeta_0^{\rm he}(\pi_t\alpha) - i_{g_t^{-1}\alpha}\zeta_0^{\rm db}(\pi_t g_t X)\right)
\end{aligned} \tag{20}$$

# Metric variations

- With $G_t = Z_t(F)$

$$\Lambda[g_t, A] = \tfrac{1}{2} F \wedge \star_t G_t \tag{21}$$

- Expanding $Z_t$ gives

$$\begin{aligned} \Lambda_t =& F \wedge \star_t(\zeta_0^{\text{de}}(i_{V_t}F) \wedge \tilde{V}_t) + F \wedge \star_t(\zeta_0^{\text{db}}(i_{V_t}\star_t F) \wedge \tilde{V}_t) \\ &+ F \wedge \zeta_0^{\text{he}}(i_{V_t}F) \wedge \tilde{V}_t + F \wedge \zeta_0^{\text{hb}}(i_{V_t}\star_t F) \wedge \tilde{V}_t \end{aligned} \tag{22}$$

- Metric variations yields:

$$\begin{aligned} \tau_a =& - F \wedge i_a \star G + i_a G \wedge \star F + 2 v_a (i_V G \wedge \star F - i_V F \wedge \star G) \\ &+ (i_V G \wedge i_V \star F - i_V F \wedge i_V \star G) \wedge e_a \end{aligned} \tag{23}$$

# Variational derivation of the Abraham Tensor

- In terms of $F$ and $G$ this gives the stress-energy-momentum tensor

$$T^{em} = i_a F \otimes i^a G + i_a G \otimes i^a F + \star(F \wedge \star G) g + \widetilde{V} \otimes \mathbf{s} + \mathbf{s} \otimes \widetilde{V} \tag{24}$$

where

$$\mathbf{s} = \star(\mathbf{e} \wedge \mathbf{h} \wedge \widetilde{V}) + \star(\mathbf{b} \wedge \mathbf{d} \wedge \widetilde{V})$$

$$\mathbf{e} = i_V F \qquad \text{and} \qquad \mathbf{b} = i_V \star F$$

$$F = \mathbf{e} \wedge \tilde{V} - \star(\mathbf{b} \wedge \tilde{V})$$

$$\mathbf{d} = i_V G\,, \qquad \text{and} \qquad \mathbf{h} = i_V \star G\,.$$

$$G = \mathbf{d} \wedge \tilde{V} - \star(\mathbf{h} \wedge \tilde{V})$$

- The *unit* timelike 4−velocity vector field $V$ describes the velocity of the medium

# Stress-energy-momentum Tensor Components

- The above derivation is covariant and describes arbitrary moving media relative to arbitrary moving observers in an arbitrary Einstein gravitational field.
- In an inertial (geodesic) frame in Minkowski spacetime with coordinates $(t, \vec{x})$ and the medium at rest in this frame:

$$g(\tilde{\mathbf{e}}, \tilde{\mathbf{d}}) = \vec{E} \cdot \vec{D} \quad , \quad g(\tilde{\mathbf{h}}, \tilde{\mathbf{b}}) = \vec{H} \cdot \vec{B}$$

  and

$$\tilde{S} = -(\vec{E} \times \vec{H}) \cdot d\vec{x}$$

- The coordinate components of this stress-energy-momentum tensor are:

$$\begin{aligned} T_{00} &= \tfrac{1}{2}(\vec{E} \cdot \vec{D} + \vec{H} \cdot \vec{B}) \\ T_{ij} &= -\tfrac{1}{2}(E_i D_j + E_j D_i) - \tfrac{1}{2}(H_i B_j + B_j H_i) + \tfrac{1}{2}\delta_{ij}(\vec{E} \cdot \vec{D} + \vec{H} \cdot \vec{B}) \\ T_{0k} &= T_{k0} = -(\vec{E} \times \vec{H})_k \end{aligned} \tag{25}$$

# Representation in terms of "Magnetisation" and "Polarisation" Forms

- Introduce the spatial polarisation $1-$form $\mathbf{p}$ and spatial magnetisation $1-$form $\mathbf{m}$ relative to medium 4-velocity $V$ by

$$\mathbf{d} = \mathbf{e} + \mathbf{p} \tag{26}$$

$$\mathbf{h} = \mathbf{b} - \mathbf{m} \tag{27}$$

Thus

$$G = F + \mathcal{P} \tag{28}$$

where

$$\mathcal{P} = \mathbf{p} \wedge \tilde{V} + \star(\mathbf{m} \wedge \tilde{V}). \tag{29}$$

# Representation in terms of "Magnetisation" and "Polarisation" Forms

$$\tau_c = \tau_c^{FG} + \tau_c^{\lambda} + \tau_c^{\Delta} + \tau_c^{V}$$

where

$$\begin{aligned}
2\,\tau_c^{FG} &= i_c \star G \wedge F - i_c G \wedge \star F\,, \\
2\tau_c^{\lambda} &= V_c \left( \mathbf{p} \wedge \star F + \mathbf{m} \wedge F - (\mathbf{p} \wedge \mathbf{b} + \mathbf{m} \wedge \mathbf{e}) \wedge \tilde{V} \right) \\
2\tau_c^{\Delta} &= -\,(i_c F) \wedge (\mathbf{m} \wedge \tilde{V}) - (i_c \star F) \wedge \star(\mathbf{m} \wedge \tilde{V}) \\
2\tau_c^{V} &= -\,V_c\,(\mathbf{p} \wedge \star F + \mathbf{m} \wedge F) \\
&\quad -\,V_c\,\tilde{V} \wedge (\mathbf{p} \wedge \mathbf{b} + \mathbf{m} \wedge \mathbf{e}) - e_c \wedge (\mathbf{p} \wedge \mathbf{b} + \mathbf{m} \wedge \mathbf{e})
\end{aligned}$$

# A Pondermotive 4-Force

- If the total stress-energy-momentum tensor $T^{tot} \equiv T^{em} + T^{m}$ then $\nabla \cdot T^{tot} = 0$ yields the dynamical field system:

$$\nabla \cdot T^{m} = \mathcal{F}^{em}$$

- where the em-pondermotive 4-force $\mathcal{F}^{em}$ for this split of $T^{tot}$ is defined as

$$\mathcal{F}^{em} \equiv -\nabla \cdot T^{em}$$

- For a cold material with internal energy density $\rho c^2$ and spatial "pressure" tensor $\mathcal{P}$

$$T^{m} \equiv \rho c^2 \tilde{V} \otimes \tilde{V} + \mathcal{P}$$

- Electro- and Magneto-striction arises from $\nabla \cdot \mathcal{P}$ when $Z$ depends on the deformation gradient of a material that can sustain elastic stress.

# Conclusions

- The long running ($\simeq$ 1900-2000) historic controversy about the "momentum of light" in matter is an argument about definitions.
- Both theoretical and experimental attempts to resolve issues have often been clouded by ambiguities in detail and lack of precision.
- A clear consistent covariant description of a system of "polarisable" matter in interaction with an electromagnetic field can be given in terms of a variational approach in spacetime.
- At the phenomenological level, this inevitably involves a spacetime tensorial approach to the constitutive properties of any medium, an assumption about the dependence of the total action on the metric and connection (on the bundle of linear frames) and hence the response of matter to gravitation.
- For matter interacting with Einstein gravitation the total stress-energy-momentum tensor defined as a source for gravitation is symmetric.

# Conclusions

- Partial stress-energy-momentum tensors need not be symmetric and may give rise to conserved quantities in regions of spacetime with Killing vectors.
- Covariant “Pondermotive forces” arise from splits of $\nabla \cdot T^{tot} = 0$.
- These conclusions have been explicitly demonstrated for the first time by considering the variations of the particular action $\int F \wedge \star G$ for a inhomogeneous anisotropic linear non-dispersive intrinsically magneto-electric medium in arbitrary motion.
- The results have implications for the interaction of charged particles with moving plasmas where gravitation is negligible as well as in astrophysical scenarios where gravitation is dominant.