Geometrical description of non-linear electrostatic oscillations in relativistic thermal plasmas

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July 31, 2008

Abstract

We develop a method for investigating the relationship between the shape of a 1-particle distribution and non-linear electrostatic oscillations in a collisionless plasma, incorporating transverse thermal motion. A general expression is found for the maximum sustainable electric field, and is evaluated for a particular highly anisotropic distribution.

Introduction

High-power lasers and plasmas may be used to accelerate electrons by electric fields that are orders of magnitude greater than those achievable using conventional methods [1]. An intense laser pulse is used to drive a wave in an underdense plasma and, for sufficiently large fields, non-linearities lead to collapse of the wave structure ("wave-breaking") due to sufficiently large numbers of electrons becoming trapped in the wave.

Hydrodynamic investigations of wave-breaking were first undertaken for cold plasmas [2, 3] and thermal effects were later included in non-relativistic [4] and relativistic contexts [5–7] (see [8] for a discussion of the numerous approaches). However, it is clear that the value of the electric field at which the wave breaks (the electric field's "wave-breaking limit") is highly sensitive to the details of the hydrodynamic model.

Plasmas dominated by collisions are described by a pressure tensor that does not deviate far from isotropy, whereas an intense and ultrashort laser pulse propagating through an underdense plasma will drive the plasma anisotropically over typical acceleration timescales. Thus, it is important to accommodate 3-dimensionality and allow for anisotropy when investigating wave-breaking limits. The sensitivity of the wave-breaking limit to the details of the plasma model suggests that it could depend on the anisotropy of the pressure tensor.

One method for investigating the wave-breaking limit of a collisionless anisotropic plasma is to employ the warm plasma closure of velocity moments of the 1-particle distribution f satisfying the Vlasov-Maxwell equations [7]. Successive order moments of the Vlasov equation induce an infinite hierarchy of field equations for the velocity moments of f and at each finite order the number of unknowns is greater than the number of field equations. The warm plasma closure scheme sets the number of unknowns equal to the number of field equations by assuming that the terms containing the third order centred moment are negligible relative to those including second, first and zeroth order centred moments.

Our aim is to uncover the relationship between wave-breaking and the shape of f. In general, the detailed structure of f cannot be reconstructed from a few low-order moments so we adopt a different approach based on a particular class of piecewise constant 1-particle distributions. Our choice of distribution, although somewhat artificial, reduces the Vlasov equation to that of a boundary in the unit hyperboloid bundle over spacetime. Combining the equation for the boundary with the Maxwell equations yields an integral for the wave-breaking limit in terms of the shape of the boundary.

Our approach may be considered as a multi-dimensional generalization of the 1-dimensional relativistic "waterbag" model employed in [5].

1 Vlasov-Maxwell equations

The brief summary of the Vlasov-Maxwell equations given below establishes our conventions. Further discussion of relativistic kinetic theory may be found in, for example, [9,10]. We employ the Einstein summation convention throughout and units are used in which the speed of light $c = 1$ and the permittivity of the vacuum $\varepsilon_0 = 1$. Lowercase Latin indices a, b, c run over $0, 1, 2, 3$.

Preliminary considerations

Let (x^a) be an inertial coordinate system on Minkowski spacetime $(\mathcal{M}, g)$ where x^0 is the proper time of observers at fixed Cartesian coordinates (x^1, x^2, x^3) in the laboratory. The metric tensor g has the form

$$g = \eta_{ab}\, dx^a \otimes dx^b \tag{1}$$

where

$$\eta_{ab} = \begin{cases} -1 \text{ if } a = b = 0 \\ 1 \text{ if } a = b \neq 0 \\ 0 \text{ if } a \neq b \end{cases} \tag{2}$$

Let $(x^a, \dot{x}^b)$ be an induced coordinate system on the total space $T\mathcal{M}$ of the tangent bundle $(T\mathcal{M}, \Pi, \mathcal{M})$ and in the following, where convenient, we will write x instead of x^a and $\dot{x}$ instead of $\dot{x}^b$.

We are interested in the evolution of a thermal plasma over timescales during which the motion of the ions is negligible in comparison with the motion of the electrons. We assume that the ions are at rest and distributed homogeneously in the laboratory frame. Their worldlines are trajectories of the vector field $N_{\text{ion}} = n_{\text{ion}}\partial/\partial x^0$ on $\mathcal{M}$ where n_{ion} is the constant ion number density measured in the laboratory frame. The electrons are described statistically by a 1-particle distribution $f(x, \dot{x})$ which induces a number 4-current vector field $N = N^a\partial/\partial x^a$

$$N^a(x) = \int_{\mathbb{R}^3} \dot{x}^a f(x, \dot{x})\, \frac{1}{\sqrt{1 + |\dot{\boldsymbol{x}}|^2}}\, d\dot{x}^1 d\dot{x}^2 d\dot{x}^3 \tag{3}$$

on $\mathcal{M}$, where $|\dot{\boldsymbol{x}}|^2 = (\dot{x}^1)^2 + (\dot{x}^2)^2 + (\dot{x}^3)^2$. One may write the Maxwell equations on $\mathcal{M}$ as

$$\frac{\partial F_{bc}}{\partial x^a} + \frac{\partial F_{ab}}{\partial x^c} + \frac{\partial F_{ca}}{\partial x^b} = 0, \tag{4}$$

$$\frac{\partial F^{ba}}{\partial x^b} = qN^a - qN^a_{\text{ion}}, \tag{5}$$

where F_{ab} are the components of the electromagnetic field tensor, $F^{ab} = \eta^{ac}\eta^{bd}F_{cd}$, q is the charge on the electron ($q < 0$) and (η^{ab}) is the matrix inverse of (η_{ab}). The scalar field f satisfies the Vlasov equation, which may be written

$$\dot{x}^a\left(\frac{\partial f}{\partial x^a} - \frac{q}{m} F^{b\boldsymbol{V}}_{\;\;a}\frac{\partial f}{\partial \dot{x}^b}\right) = 0 \tag{6}$$

on $T\mathcal{M}$ where $F^{b\boldsymbol{V}}_{\;\;a}$ is the vertical lift of $F^b{}_a = \eta^{bc}F_{ca}$ from $\mathcal{M}$ to $T\mathcal{M}$,

$$F^{b\boldsymbol{V}}_{\;\;a}(x, \dot{x}) = F^b{}_a(x). \tag{7}$$

Exterior formulation

In this section we recast the above using the tools of exterior differential calculus as it affords a succinct and powerful language for subsequent analysis. We make extensive use of Cartan's exterior derivative d, the exterior product $\wedge$ and the Hodge map $\star$ on differential forms (see, for example, [11, 12]).

The spacetime volume 4-form $\star 1$ is

$$\star 1 = dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \tag{8}$$

and the Maxwell equations (4, 5) can be written

$$dF = 0, \qquad d\star F = -q\star\widetilde{N} + q\star\widetilde{N_{\text{ion}}} \tag{9}$$

where $F = \frac{1}{2}F_{ab}\, dx^a \wedge dx^b$ is the electromagnetic 2-form, and the 1-forms $\widetilde{N}$, $\widetilde{N_{\text{ion}}}$ are the metric duals of the vector fields N, N_{ion} respectively. (The metric dual $\widetilde{Y}$ of a vector field Y satisfies $\widetilde{Y}(Z) = g(Y, Z)$ for all vector fields Z.)

Introduce the vector fields L, X,

$$L = \dot{x}^a \left(\frac{\partial}{\partial x^a} - \frac{q}{m} F^{b\boldsymbol{V}}_{\ \ a} \frac{\partial}{\partial \dot{x}^b} \right), \tag{10}$$

$$X = \dot{x}^a \frac{\partial}{\partial \dot{x}^a}, \tag{11}$$

on $T\mathcal{M}$ and the 6-form ω,

$$\omega = \iota_L \, \iota_X (\star 1^{\boldsymbol{V}} \wedge \# 1) \tag{12}$$

on $T\mathcal{M}$ where ι_Y is the interior operator on forms with respect to vector Y, the 4-form $\star 1^{\boldsymbol{V}}$

$$\star 1^{\boldsymbol{V}} = dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \tag{13}$$

is the vertical lift of the spacetime volume 4-form $\star 1$ from $\mathcal{M}$ to $T\mathcal{M}$ and the 4-form $\#1$

$$\#1 = d\dot{x}^0 \wedge d\dot{x}^1 \wedge d\dot{x}^2 \wedge d\dot{x}^3 \tag{14}$$

on $T\mathcal{M}$.

The total space $\mathcal{E}$ of the sub-bundle $(\mathcal{E}, \Pi, \mathcal{M})$ of $(T\mathcal{M}, \Pi, \mathcal{M})$ is the set of timelike, future-directed, unit normalized tangent vectors on $\mathcal{M}$,

$$\mathcal{E} = \{ (x, \dot{x}) \in T\mathcal{M} \,\big|\, \varphi = 0 \text{ and } \dot{x}^0 > 0 \} \tag{15}$$

where

$$\varphi = \eta_{ab} \, \dot{x}^a \dot{x}^b + 1. \tag{16}$$

The integral (3) can be written

$$N^a(x) = \int_{\mathcal{E}_x} \dot{x}^a f \, \iota_X \#1 \tag{17}$$

where $\mathcal{E}_x = \Pi^{-1}(x)$ is the fibre of $(\mathcal{E}, \Pi, \mathcal{M})$ over $x \in \mathcal{M}$, and it can be shown that the Vlasov equation (6) can be written

$$d(f\omega) \simeq 0 \tag{18}$$

where $\simeq$ denotes equality under restriction to $\mathcal{E}$ by pull-back. Thus, it follows

$$\int_{\mathcal{B}} d(f\omega) = 0 \tag{19}$$

where $\mathcal{B}$ is a 6-dimensional region in $\mathcal{E}$ and using the generalized Stokes theorem on forms (see, for example, [12]) we obtain

$$\int_{\partial\mathcal{B}} f\omega = 0 \tag{20}$$

where $\partial\mathcal{B}$ is the boundary of $\mathcal{B}$.

Piecewise constant distributions

We consider distributions for which $f = \alpha$ is a positive constant inside a 6-dimensional region $\mathcal{U} \subset \mathcal{E}$ and $f = 0$ outside. In particular, we consider $\mathcal{U}$ to be the union over each point $x \in \mathcal{M}$ of a domain $\mathcal{W}_x$ whose boundary $\partial\mathcal{W}_x$ in $\mathcal{E}$ is topologically equivalent to the 2-sphere. Such distributions are sometimes called "waterbags" in the literature.

Choosing $\mathcal{B}$ in (20) to be a small 6-dimensional "pill-box" that intersects $\partial\mathcal{W}_x$ and taking the appropriate limit as the volume of $\mathcal{B}$ tends to zero, we recover a jump condition on $f\omega$ that leads to

$$d\lambda \wedge \omega \simeq 0 \ \text{ at } \lambda = 0 \tag{21}$$

where $\lambda = 0$ is the union over x of the boundaries $\partial\mathcal{W}_x$.

If $\lambda = 0$ is the image of the embedding map Σ,

$$\begin{aligned}\Sigma : \mathcal{M} \times S^2 &\rightarrow \mathcal{E} \\ (x, \xi) &\mapsto (x, \dot{x} = V_\xi(x)),\end{aligned} \tag{22}$$

where $\xi = (\xi^1, \xi^2)$ is a point in S^2, then it follows from (10, 11, 12) that (21) is equivalent to

$$\big(\nabla_{V_\xi}\widetilde{V_\xi} - \frac{q}{m}\iota_{V_\xi}F\big) \wedge \Omega_\xi = 0. \tag{23}$$

Here, V_ξ and Ω_ξ are families of vector fields and 2-forms on $\mathcal{M}$ respectively, defined by

$$V_\xi = V^a_\xi \frac{\partial}{\partial x^a} = (\Sigma^* \dot{x}^a)\frac{\partial}{\partial x^a}, \tag{24}$$

$$\Omega_\xi = \frac{\partial \Sigma^* \dot{x}^a}{\partial \xi^1} dx_a \wedge \frac{\partial \Sigma^* \dot{x}^b}{\partial \xi^2} dx_b. \tag{25}$$

where $dx_a = \eta_{ab} dx^b$. Note that since the image of Σ lies in $\mathcal{E}$, it follows that, for each $\xi \in S^2$, V_ξ is timelike, unit normalized and future-directed:

$$g(V_\xi, V_\xi) = -1, \qquad g(V_\xi, \frac{\partial}{\partial x^0}) < 0. \tag{26}$$

We adopt (23) as the equation of motion for $\partial\mathcal{W}_x$.

It may be shown that a particular class of solutions to (23) satisfies

$$F = \frac{m}{q} d\widetilde{V_\xi} \tag{27}$$

and using (9) we obtain the field equation

$$d \star d\widetilde{V_\xi} = -\frac{q^2}{m}(\star\widetilde{N} - \star\widetilde{N_{\rm ion}}) \tag{28}$$

on $\mathcal{M}$ with the condition that $d\widetilde{V_\xi}$ is independent of ξ. For simplicity, we have neglected the direct contribution of the laser pulse to the total electromagnetic field in (27).

2 Electrostatic oscillations

Before analysing (28, 26) further it is useful to briefly discuss their analogue on 2-dimensional spacetime for facilitating comparison with the approach adopted in [5].

Electrostatic oscillations in 1 spatial dimension

Although formulated on 4-dimensional spacetime, equations (28, 26) have a similar structure for any number of dimensions. In particular, we now consider 2-dimensional Minkowski spacetime $(\mathcal{M}, g)$

$$g = -dt \otimes dt + dz \otimes dz, \tag{29}$$

$$\star 1 = dt \wedge dz \tag{30}$$

where (t, z)[1] is a Cartesian coordinate system in the laboratory inertial frame. An induced coordinate system on $T\mathcal{M}$ is $(t, z, \dot{t}, \dot{z})$ and note that in this sub-section of the article the fibre space of $(\mathcal{E}, \Pi, \mathcal{M})$ is 1-dimensional, whereas in the rest of the article it is 3-dimensional. Furthermore, ξ is now an element of the 0-sphere $\{+, -\}$ and $\Omega_\xi = 1$ is a constant 0-form. Thus, the analogue to (23) is

$$\begin{aligned}&\nabla_{V_+}\widetilde{V_+} - \frac{q}{m}\iota_{V_+}F = 0, \\ &\nabla_{V_-}\widetilde{V_-} - \frac{q}{m}\iota_{V_-}F = 0,\end{aligned} \tag{31}$$

[1] We use (t, z) rather than (x^a) to distinguish coordinates on 2- and 4-dimensional spacetimes.

where $V_\pm$ satisfy the conditions

$$g(V_+, V_+) = -1, \qquad g(V_+, \frac{\partial}{\partial t}) < 0,$$
$$g(V_-, V_-) = -1, \qquad g(V_-, \frac{\partial}{\partial t}) < 0, \tag{32}$$

and the only non-trivial Maxwell equation for the 2-form F is

$$d \star F = -q \star \widetilde{N} + q \star \widetilde{N_{\rm ion}} \tag{33}$$

where $N_{\rm ion} = n_{\rm ion}\partial/\partial t$ is the ion number 2-current and $F = Edt \wedge dz$ where E is the electric field along the z-axis.

On $\mathcal{E}$, $\dot{t} = \sqrt{1+\dot{z}^2}$ and the components of the electron number 2-current $N = N^t\partial/\partial t + N^z\partial/\partial z$ corresponding to (17) are

$$N^t = \int_{\mathbb{R}} f(t,z,\dot{t},\dot{z})\, d\dot{z} = \alpha\Big(X_+ - X_-\Big),$$
$$N^z = \int_{\mathbb{R}} \frac{\dot{z}}{\sqrt{1+\dot{z}^2}} f(t,z,\dot{t},\dot{z})\, d\dot{z} = \alpha\Big(\sqrt{1+X_+^2} - \sqrt{1+X_-^2}\Big), \tag{34}$$

where

$$f = \begin{cases} \alpha, & X_- \le \dot{z} \le X_+, \\ 0, & \dot{z} < X_- \text{ or } \dot{z} > X_+ \end{cases} \tag{35}$$

with α a positive constant and $\{X_+, X_-\}$ scalar fields over spacetime. The 2-velocity fields $\{V_+, V_-\}$ satisfy

$$V_\pm = \sqrt{1+X_\pm^2}\,\frac{\partial}{\partial t} + X_\pm \frac{\partial}{\partial z} \tag{36}$$

and it follows

$$\widetilde{N} = \alpha\star\big(\widetilde{V_+} - \widetilde{V_-}\big). \tag{37}$$

Unlike their 4-dimensional analogue, which may include transverse electromagnetic fields, (31) are *uniquely*[2] solved by

$$d\widetilde{V_\pm} = \frac{q}{m}F \tag{38}$$

and using (33)

$$d \star d\widetilde{V_\pm} = -\frac{q^2}{m}(\star\widetilde{N} - \star\widetilde{N_{\rm ion}}) \tag{39}$$

subject to the condition $d\widetilde{V_+} = d\widetilde{V_-}$.

Alternatively, one may follow the approach adopted in [5] employing a warm fluid model:

$$(\rho + p)\nabla_U \widetilde{U} = qn\iota_U F - \iota_U(dp \wedge \widetilde{U}), \tag{40}$$
$$g(U,U) = -1, \tag{41}$$
$$g(U, \frac{\partial}{\partial t}) < 0. \tag{42}$$

Here,

$$N = nU \tag{43}$$

where U is the bulk 2-velocity of the electron fluid,

$$U = \frac{1}{\sqrt{-g(Z,Z)}}Z, \qquad Z = \frac{1}{2}(V_+ + V_-), \tag{44}$$

[2]Proper incorporation of transverse fields requires at least 2 spatial dimensions.

and, in the electron fluid's rest frame, ρ is the fluid's energy density and p is the fluid's pressure defined as

$$\rho = \int_{\mathbb{R}} \sqrt{1+\dot{z}^2} f(t,z,\dot{t},\dot{z})\, d\dot{z} \tag{45}$$

$$p = \int_{\mathbb{R}} \frac{\dot{z}^2}{\sqrt{1+\dot{z}^2}} f(t,z,\dot{t},\dot{z})\, d\dot{z}. \tag{46}$$

It may be shown

$$\rho = m\alpha\Bigg[\frac{n}{2\alpha}\sqrt{1+\Big(\frac{n}{2\alpha}\Big)^2} + \sinh^{-1}\Big(\frac{n}{2\alpha}\Big)\Bigg], \tag{47}$$

$$p = m\alpha\Bigg[\frac{n}{2\alpha}\sqrt{1+\Big(\frac{n}{2\alpha}\Big)^2} - \sinh^{-1}\Big(\frac{n}{2\alpha}\Big)\Bigg]. \tag{48}$$

Thus, (31, 32) may be replaced by an equivalent field theory expressed in terms of a finite set of moments of f on 2-dimensional spacetime. However, the situation is more complicated for waterbags over 4-dimensional spacetime where the moment hierarchy is not automatically closed.

We will now use (39) to obtain a non-linear oscillator describing 1-dimensional electrostatic oscillations. Let all field components with respect to the laboratory frame (dt, dz) be functions of $\zeta = z - vt$ only (the "quasi-static assumption"), where $0 < v < 1$, and let (e^1, e^2) be the basis

$$e^1 = v dz - dt, \qquad e^2 = dz - v dt. \tag{49}$$

The coframe $(\gamma e^1, \gamma e^2)$ is an orthonormal basis adapted to observers moving at velocity v along z (i.e observers in the "wave frame") where $\gamma = (1-v^2)^{-1/2}$ is the Lorentz factor of such observers relative to the laboratory. For example, $\gamma e^2(N_{\text{ion}}) = -\gamma n_{\text{ion}} v$ is the ion 1-current in the wave frame.

In the basis (e^1, e^2), $\widetilde{V_\pm}$ can be decomposed as

$$\widetilde{V_\pm} = \big(\mu(\zeta) + A_\pm\big)e^1 + \psi_\pm(\zeta)e^2. \tag{50}$$

Note that this is the most general decomposition compatible with equation (38) and the quasi-static assumption.

Solving (32) for $\psi_\pm^2$ gives

$$\psi_\pm^2 = (\mu + A_\pm)^2 - \gamma^2 \tag{51}$$

and additional physical information is needed to fix the sign of $\psi_\pm$. Here, we demand that all electrons described by the waterbag are travelling slower than the wave so $\psi_\pm = -\sqrt{(\mu+A_\pm)^2 - \gamma^2}$ and (50) is

$$\widetilde{V_\pm} = (\mu + A_\pm)e^1 - \Big((\mu + A_\pm)^2 - \gamma^2\Big)^{1/2} e^2. \tag{52}$$

Substituting (50) into equation (38) yields

$$E = \frac{1}{\gamma^2}\frac{m}{q}\frac{d\mu}{d\zeta}, \tag{53}$$

and equation (39) yields the nonlinear oscillator equation

$$\frac{1}{\gamma^2}\frac{d^2\mu}{d\zeta^2} = -\frac{q^2}{m}\gamma^2 n_{\text{ion}} - \frac{q^2}{m}\alpha\Big[\sqrt{(\mu+A_+)^2-\gamma^2} - \sqrt{(\mu+A_-)^2-\gamma^2}\Big] \tag{54}$$

with the algebraic constraint

$$A_+ - A_- = -\frac{n_{\text{ion}}\gamma^2 v}{\alpha}. \tag{55}$$

Longitudinal electrostatic oscillations in 3 spatial dimensions

We now consider electrostatic waves in 3 spatial dimensions by closely following the above description of 1 dimensional electric waves.

To proceed further we seek a form for $\mathcal{W}_x$ axisymmetric about $\dot{x}^3$ whose pointwise dependence in $\mathcal{M}$ is on the wave's phase $\zeta = x^3 - vx^0$ only, where $0 < v < 1$. As before, the following results are applicable only if the longitudinal component of V_ξ in the wave frame is negative (no electron described by $\mathcal{W}_x$ is moving faster along x^3 than the wave).

Decompose $\widetilde{V}_\xi$ in the wave frame as

$$\widetilde{V}_\xi = [\mu(\zeta) + A(\xi^1)]\, e^1 + \psi(\xi^1, \zeta)\, e^2 \; + R\sin(\xi^1)\cos(\xi^2)dx^1 + R\sin(\xi^1)\sin(\xi^2)dx^2 \tag{56}$$

for $0 < \xi^1 < \pi$, $0 \leq \xi^2 < 2\pi$ where $R > 0$ is constant and

$$e^1 = vdx^3 - dx^0, \qquad e^2 = dx^3 - vdx^0. \tag{57}$$

Here, $(\gamma e^1, \gamma e^2, dx^1, dx^2)$ is an orthonormal basis (the wave frame) with $\gamma = 1/\sqrt{1-v^2}$. In the wave frame the relativistic energy of $P_\xi = mV_\xi$ is $m(\mu + A)/\gamma$ and it follows that $\mu + A > 0$. The component ψ is determined using (26),

$$\psi = -\sqrt{[\mu + A]^2 - \gamma^2[1 + R^2\sin^2(\xi^1)]}, \tag{58}$$

where the negative square root is chosen because no electron is moving faster along x^3 than the wave.

Substituting (56) into equation (27) leads to

$$F = \frac{m}{q}\frac{d\mu}{d\zeta} e^2 \wedge e^1, \tag{59}$$

and (28, 17, 56, 58) yield

$$\frac{1}{\gamma^2}\frac{d^2\mu}{d\zeta^2} = -\frac{q^2}{m} n_{\rm ion}\gamma^2 - \frac{q^2}{m} 2\pi R^2 \alpha \int_0^\pi \bigg([\mu + A(\xi^1)]^2 \; - \gamma^2[1 + R^2\sin^2(\xi^1)] \bigg)^{1/2} \sin(\xi^1)\,\cos(\xi^1)\, d\xi^1 \tag{60}$$

(c.f. equation (54)) and

$$2\pi R^2 \int_0^\pi A(\xi^1)\,\sin(\xi^1)\,\cos(\xi^1)\,d\xi^1 = -\frac{n_{\rm ion}\gamma^2\, v}{\alpha} \tag{61}$$

(c.f. equation (55)) where α is the value of f inside $\mathcal{W}_x$.

The form of the 2nd order autonomous non-linear ordinary differential equation (60) for μ is fixed by specifying the generator $A(\xi^1)$ of $\partial\mathcal{W}_x$ subject to the normalization condition (61).

3 Electrostatic wave-breaking

The form of the integrand in (60) ensures that the magnitude of oscillatory solutions to (60) cannot be arbitrarily large. For our model, the wave-breaking value $\mu_{\rm wb}$ is the largest μ for which the argument of the square root in (60) vanishes,

$$\mu_{\rm wb} = \max\bigg\{ - A(\xi^1) + \gamma\sqrt{1 + R^2\sin^2(\xi^1)} \,\bigg|\, 0 \leq \xi^1 \leq \pi \bigg\}, \tag{62}$$

because $\mu < \mu_{\rm wb}$ yields an imaginary integrand in (60) for some ξ^1. The positive square root in (62) is chosen because, as discussed above, $\mu + A(\xi^1) > 0$ and in particular $\mu_{\rm wb} + A(\xi^1) > 0$.

The electric field has only one non-zero component E (in the x^3 direction). Using $F = E\, dx^0 \wedge dx^3$ and (56, 57, 59) it follows

$$E = \frac{m}{q}\frac{1}{\gamma^2}\frac{d\mu}{d\zeta} \tag{63}$$

and the wave-breaking limit $E_{\max}$ is obtained by evaluating the first integral of (60) between $\mu_{\rm wb}$ where E vanishes and the equilibrium[3] value $\mu_{\rm eq}$ of μ where E is at a maximum. Using (61) to eliminate α it follows that $\mu_{\rm eq}$ satisfies

$$\frac{1}{v}\int_0^\pi A(\xi^1)\sin(\xi^1)\cos(\xi^1)\,d\xi^1 = \int_0^\pi \Big([\mu_{\rm eq} + A(\xi^1)]^2 - \gamma^2[1 + R^2\sin^2(\xi^1)]\Big)^{1/2}\sin(\xi^1)\cos(\xi^1)d\xi^1 \tag{64}$$

with

$$\int_0^\pi A(\xi^1)\sin(\xi^1)\cos(\xi^1)\,d\xi^1 \; < \; 0 \tag{65}$$

since $\alpha, v > 0$. Equation (60) yields the maximum value $E_{\max}$ of E,

$$E^2_{\max} = 2mn_{\rm ion}\Bigg[-\mu_{\rm eq} + \mu_{\rm wb} + \frac{v}{\int_0^\pi A(\xi^{1\prime})\sin(\xi^{1\prime})\cos(\xi^{1\prime})d\xi^{1\prime}}\times \int_{\mu_{\rm wb}}^{\mu_{\rm eq}}\int_0^\pi \Big([\mu + A(\xi^1)]^2 - \gamma^2[1 + R^2\sin^2(\xi^1)]\Big)^{1/2}\sin(\xi^1)\cos(\xi^1)d\xi^1\,d\mu\Bigg]. \tag{66}$$

Example

The above may be used to determine a wave-breaking limit for a nearly cold plasma whose distribution's transverse extent is much larger than its longitudinal extent.

Let $A(\xi^1) = -a\cos(\xi^1)$ where a is a positive constant. Using (66) it follows

$$E^2_{\max} = 2mn_{\rm ion}\Bigg[-\mu_{\rm eq} + \mu_{\rm wb} + \frac{3}{2}\frac{v}{a}\int_{\mu_{\rm wb}}^{\mu_{\rm eq}}\int_{-1}^{1}\Big([\mu + a\chi]^2 - \gamma^2[1 + R^2(1-\chi^2)]\Big)^{1/2}\chi\,d\chi\,d\mu\Bigg] \tag{67}$$

where $\chi = -\cos(\xi^1)$ and equation (64) yields

$$\frac{3}{2}\frac{v}{a}\int_{-1}^{1}\Big([\mu_{\rm eq} + a\chi]^2 - \gamma^2[1 + R^2(1-\chi^2)]\Big)^{1/2}\chi\,d\chi = 1. \tag{68}$$

Equation (62) may be written

$$\mu_{\rm wb} = \max\Big\{-a\chi + \gamma\sqrt{1 + R^2(1-\chi^2)}\;\Big|\; -1 \le \chi \le 1\Big\}, \tag{69}$$

and for a, R, γ satisfying

$$\frac{a}{R}\sqrt{\frac{1+R^2}{a^2+\gamma^2R^2}} < 1, \qquad R > 0 \tag{70}$$

the maximum of $h(\chi) = -a\chi + \gamma\sqrt{1 + R^2(1-\chi^2)}$ over $-1 \le \chi \le 1$ coincides with a turning point $\chi = \hat\chi = -\cos(\hat\xi^1)$ of h where

$$\cos(\hat\xi^1) = \frac{a}{R}\sqrt{\frac{1+R^2}{a^2+\gamma^2R^2}}. \tag{71}$$

[3] Note that the equilibrium of μ need not coincide with the plasma's thermodynamic equilibrium.

During the maximum amplitude oscillation the points $\xi^1 = \hat{\xi}^1$ catch up with the wave and it follows

$$\mu_{\rm wb} = \frac{1}{R}\sqrt{(1+R^2)(a^2+\gamma^2 R^2)}. \tag{72}$$

For $a \ll R \ll 1$ equations (67, 68, 72) yield

$$E^2_{\rm max} \approx \frac{2m^2c^2\omega_p^2}{q^2}\left(\gamma - 1 - \frac{3}{4}\frac{v}{c}\gamma R\right) \tag{73}$$

where $mc\omega_p\sqrt{2(\gamma-1)}/|q|$ is the usual relativistic cold plasma wave-breaking limit of E (see, for example, [13]) and $\omega_p = \sqrt{n_{\rm ion}q^2/(m\varepsilon_0)}$ is the plasma angular frequency. Note that the speed of light c and the permittivity ε_0 of the vacuum have been restored. The parameter R may be eliminated in favour of an effective transverse "temperature" $T_{\perp\rm eq}$ defined as

$$T_{\perp\rm eq} = \frac{1}{2k_B\, n_{\rm ion}}(P^{11}_{\rm eq} + P^{22}_{\rm eq}), \tag{74}$$

$$P^{ab}_{\rm eq} = m\alpha\int_{\mathcal{W}_{\rm eq}} \dot{x}^a\dot{x}^b \iota_X \#1 \tag{75}$$

where $\mathcal{W}_{\rm eq}$ is the support of the distribution with $\mu = \mu_{\rm eq}$ (see footnote 3) and k_B is Boltzmann's constant. It follows

$$R \approx \sqrt{\frac{5k_BT_{\perp\rm eq}}{mc^2}} \tag{76}$$

where the speed of light c has been restored.

Conclusion

We have developed a method for investigating the relationship between the shape of a 1-particle distribution and electrostatic non-linear thermal plasma waves near breaking. An approximation to the wave-breaking limit of the electric field was obtained for a particular axisymmetric distribution.

Further analysis of (66, 64, 62) will be presented elsewhere.

Acknowledgements

We thank RA Cairns, B Ersfeld, A Reitsma and RMGM Trines for useful discussions. This work is supported by EPSRC grant EP/E022995/1.

References

[1] T Tajima and JM Dawson, Phys. Rev. Lett. 43 (1979) 267

[2] AI Akhiezer and RV Polovin, Sov. Phys. JETP 3 (1956) 696

[3] JM Dawson, Phys. Rev. 113 (1959) 383

[4] TP Coffey, Phys. Fluids 14 (1971) 1402

[5] T Katsouleas and WB Mori, Phys. Rev. Lett. 61 (1988) 90

[6] JB Rosenzweig, Phys. Rev. A 38 (1988) 3634

[7] CB Schroeder, E Esarey and BA Shadwick, Phys. Rev. E 72 (2005) 055401

[8] RMGM Trines and PA Norreys, Phys. Plasmas 13 (2006) 123102

[9] J Ehlers in *General Relativity and Cosmology*, Proceedings of the International School of Physics "Enrico Fermi" 47, (Academic Press, New York and London, 1971) 1

[10] SR de Groot, WA Leeuwen and CG van Weert, *Relativistic Kinetic Theory* (North-Holland, Amsterdam, 1980)

[11] DA Burton, Theoret. Appl. Mech. 30 (2003) 85

[12] IM Benn and RW Tucker, *An Introduction to Spinors and Geometry with Applications in Physics* (Adam Hilger, Bristol and New York, 1987)

[13] WB Mori and T Katsouleas, Phys. Scr. T30 (1990) 127