On the effects of geometry on guided electromagnetic waves

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Abstract

The method of moving (Cartan) coframes is used to analyse the influence of geometry on the behaviour of electromagnetic fields in confining guides and the effect of such fields on their ultra-relativistic sources. Such issues are of relevance to a number of topical problems in accelerator science where the need to control the motion of high current-density micro-meter size bunches of relativistic radiating charge remains a technical and theoretical challenge. By dimensionally reducing the exterior equations for the sources and fields on spacetime using symmetries exhibited by the confining guides one achieves a unifying view that offers natural perturbative approaches for dealing with smooth non-uniform and curved guides. The issue of the back-reaction of radiation fields on the sources is approached in terms of a simple charged relativistic fluid model.[1]

1 Introduction

This article outlines procedures for analysing theoretically the behaviour of electromagnetic fields and their sources in metallic guides. The emphasis is on deriving systems of equations, based on Maxwell's theory, from first principles so that approximation schemes can be put into some kind of perspective. The motivation is to formulate a methodology that is powerful enough to accommodate dynamic currents i.e. describe the

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behaviour of currents produced by moving charges in guides taking into account the back-reaction on the source produced by the excited electromagnetic field. Such issues have direct relevance to many problems of interest in current particle accelerator design.

The natural mathematical language for relativistic physical field theories is the exterior calculus. Elementary introductions to this geometrical formulation can be found in [1], [2], [3], [4]. It is the author's hope that the power of this mathematical framework will be illuminated by its application to the well trodden path under discussion. In much of the literature, however, this path often stops at points where the physics of back-reaction on sources becomes interesting. Although back-reaction is arguably negligible in many applications (particulary in low power microwave transmission), this may not be the case for the accelerator scientist struggling to control the motion of bunches of electrons that are emitting copious synchrotron radiation. It is then crucial to determine how the self-consistent electromagnetic field affects such bunches and their neighbours. To deal with this interaction while maintaining the appropriate boundary conditions on the fields in the guide is a non-trivial mathematical problem and recourse to numerical simulation or approximation schemes becomes essential. For sources composed of ultra-relativistic charged particles it proves useful to re-formulate the equations of motion for the sources and electromagnetic fields into their "longitudinal"and "transverse"components. The Maxwell system can then be recast into a set of equations that resembles a collection of "telegraph"equations that determines the Maxwell fields. Indeed approximations to these equations involving lumped impedances for long transmission lines were guessed by Kelvin and others even before Maxwell invented his field theory of electromagnetism! An advantage of this formulation is that, besides ensuring the satisfaction of boundary conditions, it offers different approximation schemes for dealing with coupled sources and irregularities in the geometry of the guide in terms of a $2-$dimensional field theory rather than that based on the original set of partial differential equations in four independent variables. This approach is afforded by the notion of dimensional reduction and the exploitation of spacetime symmetries that lies behind much of the following.

2 Notation

The notation follows standard conventions for a manifold M with a metric tensor field. Thus ΓTM denotes the set of vector fields and $\Gamma\Lambda^p M$ the set of $p-$form fields on M. Metric duals with respect to any metric tensor g are written with a tilde so that $\widetilde{X} = g(X, -) \in \Gamma\Lambda^1 M$ for $X \in \Gamma TM$ and $\widetilde{\alpha} = g^{-1}(\alpha, -) \in \Gamma TM$ for $\alpha \in \Gamma\Lambda^1 M$. The Hodge dual map associated with g is denoted by a star so that the canonical n-form measure on M is the image of 1 under the Hodge map. In this article a number of different metrics will be introduced on manifolds of different dimensions. One must then distinguish notationally between the different metrics introduced and their associated Hodge maps. However for any manifold M with Hodge map $\star$ one always has the standard relations

$$\Phi \wedge \star\Psi = \Psi \wedge \star\Phi \qquad \text{for} \quad \Phi, \Psi \in \Gamma\Lambda^p M \tag{1}$$

$$i_X \star \Phi = \star(\Phi \wedge \tilde{X}) \qquad \text{for} \quad X \in \Gamma TM,\ \Phi \in \Gamma\Lambda^p M \tag{2}$$

where i_X denotes the interior (contraction) operator on forms. Maxwell's equations find their most cogent formulation as a theory of $2-$forms on spacetime modelled on a space and time oriented $4-$dimensional manifold with a metric tensor field g of Lorentzian signature $(-, +, +, +)$. On a spacetime M the set $\{e^0, e^1, e^2, e^3\}$ will denote a local g-orthonormal coframe (a linearly independent collection of $1-$forms). The Hodge map associated with the Lorentzian metric g will be denoted by $\star$. Then

$$\star i_X \Phi = - \star \Phi \wedge \tilde{X} \qquad \text{for} \quad X \in \Gamma TM,\ \Phi \in \Gamma\Lambda^p M \tag{3}$$

$$\star \star \Phi = (-1)^{p+1}\Phi \qquad \text{for} \quad \Phi \in \Gamma\Lambda^p M \tag{4}$$

For manifolds with a Euclidean signature and different dimensions these last two relations change as will be indicated for 3 and 2 dimensional spaces below. Finally note that for all $n-$dimensional manifolds of any signature one has the useful results:

$$i_X\Phi \wedge \Psi = (-1)^{p+1}\Phi \wedge i_X\Psi \quad \text{for} \quad \Phi \in \Gamma\Lambda^p M,\ \Psi \in \Gamma\Lambda^q M,\ p+q \geq n+1 \qquad (5)$$

$$d\Phi \wedge \Psi = (-1)^{p+1}\Phi \wedge d\Psi + d(\Phi \wedge \Psi) \quad \text{for} \quad \Phi \in \Gamma\Lambda^p M,\ \Psi \in \Gamma\Lambda^q M \qquad (6)$$

3 Electromagnetic fields in spacetime

Maxwell's equations for an electromagnetic field in an arbitrary medium can be written

$$d\,F = 0 \qquad \text{and} \qquad d\star G = j \qquad (7)$$

where $F \in \Gamma\Lambda^2 M$ is the Maxwell 2-form, $G \in \Gamma\Lambda^2 M$ is the excitation 2-form and $j \in \Gamma\Lambda^3 M$ is the 3-form electric current source[2]. To close this system, "electromagnetic constitutive relations" relating G and j to F are necessary.

The electric 4-current j describes both (mobile) electric charge and effective (Ohmic) currents in a conducting medium. The *electric* field $\mathbf{e} \in \Gamma\Lambda^1 M$ and *magnetic induction field* $\mathbf{b} \in \Gamma\Lambda^1 M$ associated with F are defined with respect to an arbitrary *unit* future-pointing timelike 4−velocity vector field $U \in \Gamma TM$ by

$$\mathbf{e} = i_U F \qquad \text{and} \qquad c\,\mathbf{b} = i_U\star F \qquad (8)$$

Thus $i_U\mathbf{e} = 0$ and $i_U\mathbf{b} = 0$.

Since $g(U,U) = -1$

$$F = \mathbf{e} \wedge \widetilde{U} - \star(\,c\,\mathbf{b} \wedge \widetilde{U}) \qquad (9)$$

The field U may be used to describe an *observer frame* on spacetime and its integral curves model idealised observers.

[2]All tensors in this article have dimensions constructed from the SI dimensions $[M], [L], [T], [Q]$ where $[Q]$ has the unit of the Coulomb in the MKS system. We adopt $[g] = [L^2], [G] = [j] = [Q]$, $[F] = [Q]/\epsilon_0$ where the permittivity of free space ϵ_0 has the dimensions $[Q^2\, T^2 M^{-1}\, L^{-3}]$ and c denotes the speed of light in vacuo

Likewise the *displacement* field $\mathbf{d} \in \Gamma\Lambda^1 M$ and the *magnetic* field $\mathbf{h} \in \Gamma\Lambda^1 M$ associated with G are defined with respect to U by

$$\mathbf{d} = i_U G\,, \qquad \text{and} \qquad \mathbf{h}/\,c\, = i_U \star G\,. \tag{10}$$

Thus

$$G = \mathbf{d} \wedge \widetilde{U} - \star((\mathbf{h}/\,c\,) \wedge \widetilde{U}) \tag{11}$$

and $i_U \mathbf{e} = 0$ and $i_U \mathbf{b} = 0$. It may be assumed that a material medium has associated with it a future-pointing timelike unit vector field V which may be identified with the bulk $4-$velocity field of the medium in spacetime. Integral curves of V define the averaged world-lines of identifiable constituents of the medium. A *comoving observer frame with* $4-$*velocity* U will have [3] $U = V$.

4 Time dependent Maxwell systems in space

On any $n-$dimensional manifold a chart sets up a correspondence between points on some region (patch) on the manifold and a set on $\mathbb{R}^n$. Thus in a $2-$dimensional patch let $\hat{\underline{\xi}} = (\xi^1, \xi^2)$ be a generic set of coordinates. Similarly let $\underline{\xi} = (\xi^1, \xi^2, \xi^3)$ denote coordinates on a patch of a $3-$dimensional manifold and $\xi = (\underline{\xi}, \xi^0)$ denote coordinates on a patch of $4-$dimensional spacetime.

Let $\boldsymbol{d}$ denote exterior differentiation in a patch with coordinates $\underline{\xi}$. Similarly let $\hat{\boldsymbol{d}}$ denote exterior differentiation in a patch with coordinates $\hat{\underline{\xi}}$. A "moving"orthonormal (Cartan) coframe in flat spacetime with Minkowski metric g is a set of (independent [4]) 1-forms $\{e^0, e^1, e^2, e^3\}$ with e^0 timelike. In general this will depend on the choice of coordinates ξ in the sense that their exterior derivatives will not be zero. In the following we adopt an inertial frame with laboratory time $\xi^0 = t$ and $e^0 = c\,dt$ with $\{e^1, e^2, e^3\}$ independent of t. Thus in general the coframe "moves"as a

[3] If $U \neq V$ but at an event p in spacetime their integral curves share the same tangent then it is sometimes said that V is *instantaneously* at rest at p with respect to the timelike frame U

[4] i.e. $e^0 \wedge e^1 \wedge e^2 \wedge e^3 \neq 0$

function of $\underline{\xi}$. If β is any form on spacetime it will be convenient to adopt the abbreviation $\dot{\beta}$ for $\mathcal{L}_{\frac{\partial}{\partial t}}\beta$, where $\mathcal{L}_X$ denotes the Lie derivative [3] with respect to X. Within this framework introduce the tensors:

$$\hat{g} = e^1 \otimes e^1 + e^2 \otimes e^2, \quad \underline{g} = \hat{g} + e^3 \otimes e^3, \quad g = -e^0 \otimes e^0 + \underline{g}$$

where $e^0 = c\,dt$ and g is the metric tensor on Minkowski spacetime. At each instant (t =constant), $\underline{g}$ is the induced metric tensor on Euclidean space and $\hat{g}$ is the induced metric tensor on the $2-$dimensional submanifolds (leaves) where ξ_3 = constant. Denote the Hodge map associated with $\hat{g}$ by $\hat{\#}$ with

$$\hat{\#}1 = e^1 \wedge e^2$$

Denote the Hodge map associated with $\underline{g}$ by $\#$ with

$$\#1 = \hat{\#}1 \wedge e^3$$

then

$$\star 1 = \#1 \wedge e^0 \equiv e^1 \wedge e^2 \wedge e^3 \wedge e^0$$

To accommodate the effects of signature it is convenient to introduce the involution operator η on $p-$forms Φ by $\eta\Phi = (-1)^p\Phi$. Then

$$\star\star = -\eta, \qquad \#\# = 1, \qquad \hat{\#}\hat{\#} = \eta \tag{12}$$

By linearity the action of the Hodge map on an arbitrary form in Euclidean $3-$ space readily follows by expanding it in an orthonormal basis and using the relations

$$\#e^1 = e^2 \wedge e^3$$

$$\#e^2 = e^3 \wedge e^1$$

$$\#e^3 = e^1 \wedge e^2$$

on the basis forms. Furthermore in a $2-$dimensional Euclidean space

$$\hat{\#}e^1 = e^2$$

$$\hat{\#}e^2 = -e^1$$

If $\underset{(p)}{\beta}(\xi)$ is a p-form *on spacetime* but generated by forms in the exterior algebra generated by $\{e^1(\underline{\xi}), e^2(\underline{\xi}), e^3(\underline{\xi})\}$ then at any event with coordinates ξ one has

$$\underset{(p)}{\beta}(\xi) = \sum_I \beta_I(\xi) e^I(\underline{\xi})$$

where, for each multi-index I, the set of exterior $p-$forms $\{e^I(\underline{\xi})\}$ denotes a basis for p-forms generated from the set $\{e^1(\underline{\xi}), e^2(\underline{\xi}), e^3(\underline{\xi})\}$. One refers to the functions β_I as the components of $\underset{(p)}{\beta}$ in the e^I basis. With this notation

$$\underset{(p)}{\dot{\beta}}(\xi) \equiv \sum_I \frac{\partial}{\partial \xi^0} \beta_I(\xi) e^I(\underline{\xi})$$

Define the $2+1$ split of $\underset{(p)}{\beta}(\xi)$ into the pair $\{\underset{(p-1)}{\hat{\beta}}(\xi), \underset{(p)}{\hat{\beta}}(\xi)\}$ by the unique decomposition with respect to $\boldsymbol{d}\xi^3$:

$$\underset{(p)}{\beta}(\xi) = \underset{(p-1)}{\hat{\beta}}(\xi) \wedge \boldsymbol{d}\xi^3 + \underset{(p)}{\hat{\beta}}(\xi) \tag{13}$$

where $\underset{(p-1)}{\hat{\beta}}(\xi)$ and $\underset{(p)}{\hat{\beta}}(\xi)$ are $p-1$ and p forms respectively, generated from the 1-forms in $\{\boldsymbol{d}\xi^1, \boldsymbol{d}\xi^2\}$ satisfying $i_{\frac{\partial}{\partial\xi^3}} \underset{(p-1)}{\hat{\beta}}(\xi) = 0$ and $i_{\frac{\partial}{\partial\xi^3}} \underset{(p)}{\hat{\beta}}(\xi) = 0$. Thus $\underset{(p-1)}{\hat{\beta}}$ and $\underset{(p)}{\hat{\beta}}$ are forms that do not contain $\boldsymbol{d}\xi^3$.

It follows that for $q = 0, 1, 2$:

$$\#(\underset{(q)}{\hat{\beta}} \wedge e^3) = \hat{\#}(\eta \underset{(q)}{\hat{\beta}}) \tag{14}$$

$$\#(\underset{(q)}{\hat{\beta}}) = \hat{\#}(\underset{(q)}{\hat{\beta}}) \wedge e^3 \tag{15}$$

For any $0-$form $\underset{(0)}{\hat{\beta}}$

$$\boldsymbol{d}\underset{(0)}{\hat{\beta}} = \hat{\boldsymbol{d}}\underset{(0)}{\hat{\beta}} + (\mathcal{L}_{\frac{\partial}{\partial\xi^3}} \underset{(0)}{\hat{\beta}})\, \boldsymbol{d}\xi^3$$

where

$$\hat{\boldsymbol{d}}\,\underset{(0)}{\hat{\beta}} \equiv \frac{\partial}{\partial \xi^1}\underset{(0)}{\hat{\beta}}\,\boldsymbol{d}\xi^1 + \frac{\partial}{\partial \xi^2}\underset{(0)}{\hat{\beta}}\,\boldsymbol{d}\xi^2$$

From this it follows that, for $q = 0, 1, 2$:

$$\boldsymbol{d}\,\underset{(q)}{\hat{\beta}} = \hat{\boldsymbol{d}}\,\underset{(q)}{\hat{\beta}} + \boldsymbol{d}\xi^3 \wedge (\mathcal{L}_{\frac{\partial}{\partial \xi^3}}\underset{(q)}{\hat{\beta}})$$

where $\hat{\boldsymbol{d}}$ acts on exterior forms generated by $\{\boldsymbol{d}\xi^1, \boldsymbol{d}\xi^2\}$. Note that for all 2-forms $\underset{(2)}{\hat{\beta}}$ one has $\hat{\boldsymbol{d}}\,\underset{(2)}{\hat{\beta}} = 0$. Let the 3+1 split of the 4-current $3-$form be

$$\underset{(3)}{j}(\xi) = -\underset{(2)}{J}(\xi) \wedge dt + \underset{(0)}{\rho}(\xi)\#1 \tag{16}$$

with $i_{\frac{\partial}{\partial t}}\underset{(2)}{J} = 0$. Then, from (7)

$$dj = 0 \tag{17}$$

yields

$$\boldsymbol{d}\,\underset{(2)}{J}(\xi) + \underset{(0)}{\dot{\rho}}(\xi)\#1 = 0$$

It is convenient to introduce the (Hodge) dual forms:

$$\underset{(2)}{\mathbf{E}} = \#\underset{(1)}{\mathbf{e}}, \quad \underset{(2)}{\mathbf{D}} = \#\underset{(1)}{\mathbf{d}}, \quad \underset{(2)}{\mathbf{B}} = \#\underset{(1)}{\mathbf{b}}, \quad \underset{(2)}{\mathbf{H}} = \#\underset{(1)}{\mathbf{h}}$$

so that the 3+1 split of the spacetime covariant Maxwell equations (7) with respect to dt becomes

$$\boldsymbol{d}\,\underset{(1)}{\mathbf{e}} = -\underset{(2)}{\dot{\mathbf{B}}} \tag{18}$$

$$\boldsymbol{d}\underset{(2)}{\mathbf{B}} = 0 \tag{19}$$

$$\boldsymbol{d}\,\underset{(1)}{\mathbf{h}} = \underset{(2)}{J} + \underset{(2)}{\dot{\mathbf{D}}} \tag{20}$$

$$\boldsymbol{d}\,\underset{(2)}{\mathbf{D}} = \underset{(0)}{\rho}\,\#1 \tag{21}$$

All $p-$forms ($p \geq 1$) in these equations are independent of e^0 but may depend on t. Furthermore they are independent of the choice of (stationary) spatial co-frame constructed from $\{\boldsymbol{d}\,\xi^1, \boldsymbol{d}\,\xi^2, \boldsymbol{d}\,\xi^3\}$, in any chart with local coordinates ξ^1, ξ^2, ξ^3.

In the following it is assumed that $\underset{(1)}{\mathbf{b}} = \mu\,\underset{(1)}{\mathbf{h}}$ and $\underset{(1)}{\mathbf{d}} = \epsilon\,\underset{(1)}{\mathbf{e}}$ (with constant ϵ, μ) where $\epsilon = \epsilon_r \epsilon_0$, $\mu = \mu_r \mu_0$. Thus in terms of $\underset{(1)}{\mathbf{e}}, \underset{(1)}{\mathbf{h}}, \underset{(2)}{\mathbf{E}}, \underset{(2)}{\mathbf{H}}$:

$$\boldsymbol{d}\,\underset{(1)}{\mathbf{e}} = -\mu \underset{(2)}{\dot{\mathbf{H}}} \tag{22}$$

$$\boldsymbol{d}\,\underset{(2)}{\mathbf{H}} = 0 \tag{23}$$

$$\boldsymbol{d}\,\underset{(1)}{\mathbf{h}} = \epsilon \underset{(2)}{\dot{\mathbf{E}}} + \underset{(2)}{J} \tag{24}$$

$$\epsilon \boldsymbol{d}\,\underset{(2)}{\mathbf{E}} = \underset{(0)}{\rho}\,\#1 \tag{25}$$

5 The Maxwell system inside a regular cylinder

Consider a regular hollow perfectly conducting cylinder of radius a, given in cylindrical coordinates with $\{\xi^1 \equiv r, \xi^2 \equiv \phi, \xi^3 \equiv z\}$ by $r = a$: A convenient orthonormal coframe in this coordinate system is given by

$$\{e^1 = \boldsymbol{d}\,r, \quad e^2 = r\,\boldsymbol{d}\,\phi, \quad e^3 = \boldsymbol{d}\,z\}$$

with the tensor $\hat{g} = e^1 \otimes e^1 + e^2 \otimes e^2 = (\boldsymbol{d}\,r \otimes \boldsymbol{d}\,r + r^2 \boldsymbol{d}\,\phi \otimes \boldsymbol{d}\,\phi)$. One has $\mathcal{L}_{\frac{\partial}{\partial z}} \hat{g} = 0$ and hence $\mathcal{L}_{\frac{\partial}{\partial z}} \hat{\#}1 = 0$ reflecting the translational symmetry of the regular cylinder. An important set of equations arises from taking the exterior derivative of this coframe. These *structure equations* follow simply as

$$\boldsymbol{d}\, e^1 = 0,$$
$$\boldsymbol{d}\, e^2 = \frac{1}{r}\, e^1 \wedge e^2$$
$$\boldsymbol{d}\, e^3 = 0$$

Thus for $q = 0, 1, 2$:

$$\boldsymbol{d}\,(\underset{(q)}{\hat\beta} \wedge e^3) = (\boldsymbol{d}\,\underset{(q)}{\hat\beta}\,) \wedge e^3$$

Construct the following 2+1 splits with respect to $\boldsymbol{d}\,z$

$$\underset{(1)}{\mathbf{e}} = \underset{(0)}{\hat{\mathbf{e}}} \wedge \boldsymbol{d}\,z + \underset{(1)}{\hat{\mathbf{e}}} \tag{26}$$

$$\underset{(1)}{\mathbf{h}} = \underset{(0)}{\hat{\mathbf{h}}} \wedge \boldsymbol{d}\,z + \underset{(1)}{\hat{\mathbf{h}}} \tag{27}$$

Hence

$$\underset{(2)}{\mathbf{H}} = \#\underset{(1)}{\mathbf{h}} = \hat{\#}\underset{(0)}{\hat{\mathbf{h}}} + (\hat{\#}\underset{(1)}{\hat{\mathbf{h}}}) \wedge \boldsymbol{d}\,z \tag{28}$$

and the Maxwell equation $\boldsymbol{d}\,\underset{(1)}{\mathbf{e}} = -\mu\underset{(2)}{\dot{\mathbf{H}}}$ yields:

$$\hat{\boldsymbol{d}}\,\underset{(0)}{\hat{\mathbf{e}}} \wedge \boldsymbol{d}\,z + \hat{\boldsymbol{d}}\,\underset{(1)}{\hat{\mathbf{e}}} + \boldsymbol{d}\,z \wedge \mathcal{L}_{\frac{\partial}{\partial z}}\underset{(1)}{\hat{\mathbf{e}}} = -\mu\hat{\#}\underset{(0)}{\dot{\hat{\mathbf{h}}}} - \mu\hat{\#}\underset{(1)}{\dot{\hat{\mathbf{h}}}} \wedge \boldsymbol{d}\,z$$

Projecting this with respect to $\boldsymbol{d}\,z$ gives the pair

$$\hat{\boldsymbol{d}}\,\underset{(0)}{\hat{\mathbf{e}}} - \mathcal{L}_{\frac{\partial}{\partial z}}\underset{(1)}{\hat{\mathbf{e}}} = -\mu\hat{\#}\underset{(1)}{\dot{\hat{\mathbf{h}}}} \tag{29}$$

$$\hat{\boldsymbol{d}}\,\underset{(1)}{\hat{\mathbf{e}}} = -\mu\hat{\#}\underset{(0)}{\dot{\hat{\mathbf{h}}}} \tag{30}$$

Similarly with the 2+1 split

$$\underset{(2)}{J} = \underset{(1)}{\hat{J}} \wedge \boldsymbol{d} z + \underset{(2)}{\hat{J}} \tag{31}$$

the Maxwell equation $\boldsymbol{d}\,\underset{(1)}{\mathbf{h}} = \epsilon \underset{(2)}{\dot{\mathbf{E}}} + \underset{(2)}{J}$ yields

$$\hat{\boldsymbol{d}}\,\underset{(0)}{\hat{\mathbf{h}}} - \mathcal{L}_{\frac{\partial}{\partial z}}\underset{(1)}{\hat{\mathbf{h}}} = \epsilon \hat{\#}\,\underset{(1)}{\dot{\hat{\mathbf{e}}}} + \underset{(1)}{\hat{J}} \tag{32}$$

$$\hat{\boldsymbol{d}}\,\underset{(1)}{\hat{\mathbf{h}}} = \epsilon \hat{\#}\,\underset{(0)}{\dot{\hat{\mathbf{e}}}} + \underset{(2)}{\hat{J}} \tag{33}$$

Next with the 2+1 splits

$$\underset{(2)}{\mathbf{E}} = \hat{\#}\,\underset{(0)}{\hat{\mathbf{e}}} + (\hat{\#}\,\underset{(1)}{\hat{\mathbf{e}}}) \wedge \boldsymbol{d} z \tag{34}$$

and

$$\boldsymbol{d}\,\underset{(2)}{\mathbf{E}} = \boldsymbol{d} z \wedge (\mathcal{L}_{\frac{\partial}{\partial z}}\underset{(0)}{\hat{\mathbf{e}}}\,\hat{\#}1 + \hat{\boldsymbol{d}}\,\hat{\#}\,\underset{(1)}{\hat{\mathbf{e}}}) \tag{35}$$

the Maxwell equation $\epsilon \boldsymbol{d}\,\underset{(2)}{\mathbf{E}} = \underset{(0)}{\rho}\,\#1$ yields:

$$\hat{\boldsymbol{d}}\,\hat{\#}\,\underset{(1)}{\hat{\mathbf{e}}} + \mathcal{L}_{\frac{\partial}{\partial z}}\underset{(0)}{\hat{\mathbf{e}}}\,\hat{\#}1 = \frac{1}{\epsilon}\,\underset{(0)}{\rho}\,\hat{\#}1 \tag{36}$$

Similarly the Maxwell equation $\boldsymbol{d}\,\underset{(2)}{\mathbf{H}} = 0$ yields

$$\hat{\boldsymbol{d}}\,\hat{\#}\,\underset{(1)}{\hat{\mathbf{h}}} + \mathcal{L}_{\frac{\partial}{\partial z}}\underset{(0)}{\hat{\mathbf{h}}}\,\hat{\#}1 = 0 \tag{37}$$

The 4-current conservation (compatibility) equation becomes:

$$\hat{\boldsymbol{d}}\,\underset{(1)}{\hat{J}} + \mathcal{L}_{\frac{\partial}{\partial z}}\underset{(2)}{\hat{J}} + \underset{(0)}{\dot{\rho}}\,\hat{\#}1 = 0 \tag{38}$$

In the following it proves convenient to introduce the *longitudinal current* $\underset{(0)}{\hat{J}}$ by

$$\underset{(0)}{\hat{J}} = \hat{\#}\,\underset{(2)}{\hat{J}} \tag{39}$$

6 Dirichlet modes

Let $\mathcal{D}$ be the smooth 2-dimensional submanifold (ξ^3=constant) with boundary $\partial\mathcal{D}$, embedded in Eucldean $\mathbb{R}^3$. The tensor $\hat{g}$ on $\mathcal{D}$ is that induced from the Euclidean metric $\underline{g}$ in $\mathbb{R}^3$. A real *Dirichlet mode set* $\{\Phi_N\}$ is a collection of real eigen 0−forms of the Laplacian operator $-\hat{\boldsymbol{d}}\,\hat{\#}\hat{\boldsymbol{d}}$ on $\mathcal{D}$ (associated with the metric $\hat{g}$ and Hodge operator $\hat{\#}$) that vanishes on $\partial\mathcal{D}$. This boundary condition and the nature of the domain determine the associated (positive non-zero real) eigenvalues β_N^2. The label N here consists of an ordered pair of labels. Thus

$$\Phi_N : \mathbb{R}^2 \to \mathcal{D}, \quad \hat{\underline{\xi}} \mapsto \Phi_N(\hat{\underline{\xi}}) \tag{40}$$

satisfies

$$\hat{\boldsymbol{d}}\,\hat{\#}\hat{\boldsymbol{d}}\,\Phi_N + \beta_N^2\,\Phi_N\hat{\#}1 = 0 \tag{41}$$

with $\Phi_N|_{\partial\mathcal{D}} = 0$. It is straightforward to show from these properties that if $\beta_N^2 \neq \beta_M^2 \neq 0$ then

$$\int_{\mathcal{D}} \Phi_M\,\Phi_N\,\hat{\#}1 = 0 \tag{42}$$

If one normalises these modes so that

$$\int_{\mathcal{D}} \Phi_M\,\Phi_N\,\hat{\#}1 = \mathcal{N}_N^2\,\delta_{NM} \tag{43}$$

then it is also easy to show that

$$\int_{\mathcal{D}} \hat{\boldsymbol{d}}\,\Phi_N \wedge \hat{\#}\hat{\boldsymbol{d}}\,\Phi_M = \beta_N^2\,\mathcal{N}_N^2\,\delta_{NM} \tag{44}$$

7 Neumann modes

In a similar manner one defines a real *Neumann mode set* $\{\Psi_N\}$ as a collection of real eigen 0−forms of the Laplacian operator on $\mathcal{D}$ such that the pull back to $\partial\mathcal{D}$ of $\hat{\#}\hat{\boldsymbol{d}}\ \Psi_N$ vanishes[5]. This alternative boundary condition and the nature of the domain determine the associated (positive non-zero real) eigenvalues α_N^2 where again the label N here consists of an ordered pair of real numbers. Thus

$$\Psi_N : \mathbb{R}^2 \to \mathcal{D}, \quad \hat{\underline{\xi}} \mapsto \Psi_N(\hat{\underline{\xi}}) \tag{45}$$

satisfies

$$\hat{\boldsymbol{d}}\ \hat{\#}\hat{\boldsymbol{d}}\ \Psi_N + \alpha_N^2\ \Psi_N \hat{\#}1 = 0 \tag{46}$$

with $\hat{\#}\hat{\boldsymbol{d}}\ \Psi_N|_{\partial\mathcal{D}} = 0$. It is straightforward to show from these properties that if $\alpha_N^2 \neq \alpha_M^2 \neq 0$ then

$$\int_{\mathcal{D}} \Psi_M\, \Psi_N\, \hat{\#}1 = 0 \tag{47}$$

If one normalises these modes so that

$$\int_{\mathcal{D}} \Psi_M\, \Psi_N\, \hat{\#}1 = \mathcal{M}_N^2\, \delta_{NM} \tag{48}$$

then it is also easy to how that

$$\int_{\mathcal{D}} \hat{\boldsymbol{d}}\ \Psi_N \wedge \hat{\#}\hat{\boldsymbol{d}}\ \Psi_M = \alpha_N^2\, \mathcal{M}_N^2\, \delta_{NM} \tag{49}$$

Furthermore since

$$\hat{\boldsymbol{d}}\ (\Phi_N \hat{\#}\hat{\boldsymbol{d}}\ \Psi_M) = \hat{\boldsymbol{d}}\ \Phi_N \wedge \hat{\#}\hat{\boldsymbol{d}}\ \Psi_M + \Phi_N \hat{\boldsymbol{d}}\ \hat{\#}\hat{\boldsymbol{d}}\ \Psi_M \tag{50}$$

the above properties imply that for all M, N if $\alpha_M^2 \neq \beta_N^2 \neq 0$

$$\int_{\mathcal{D}} \Psi_M\, \Phi_N\, \hat{\#}1 = \frac{1}{\alpha_N^2 - \beta_M^2} \int_{\partial\mathcal{D}} \Psi_M\, \hat{\#}\hat{\boldsymbol{d}}\ \Phi_N \tag{51}$$

[5]If the boundary $\partial\mathcal{D}$ is given by the equation $f(\underline{\xi}) = 0$ in $\mathbf{R}^3$ this is equivalent to $\boldsymbol{d} f \wedge \hat{\#}\hat{\boldsymbol{d}}\ \Psi_N = 0$

8 Source free pure cylinder harmonic modes

For the parameters k, ω consider harmonic (propagating) fields in the cylinder, free of sources j, defined by

$$\underset{(1)}{\hat{\mathbf{e}}}(r,\phi,z,t) = \underset{(1)}{\check{\mathbf{e}}}(r,\phi)e^{i(kz-\omega t)} \tag{52}$$

$$\underset{(0)}{\hat{\mathbf{e}}}(r,\phi,z,t) = \underset{(0)}{\check{\mathbf{e}}}(r,\phi)e^{i(kz-\omega t)} \tag{53}$$

$$\underset{(1)}{\hat{\mathbf{h}}}(r,\phi,z,t) = \underset{(1)}{\check{\mathbf{h}}}(r,\phi)e^{i(kz-\omega t)} \tag{54}$$

$$\underset{(0)}{\hat{\mathbf{h}}}(r,\phi,z,t) = \underset{(0)}{\check{\mathbf{h}}}(r,\phi)e^{i(kz-\omega t)} \tag{55}$$

If these fields are to exist in the guide without sources they must satisfy (29), (30), (32),(33),(36),(37). Inserting the above in these Maxwell equations yields the reduced system:

$$\hat{\boldsymbol{d}}\ \underset{(1)}{\check{\mathbf{e}}} = i\,\mu\,\omega\,\underset{(0)}{\check{\mathbf{h}}}\hat{\#}1 \tag{56}$$

$$\hat{\boldsymbol{d}}\ \underset{(1)}{\check{\mathbf{h}}} = -i\,\epsilon\,\omega\,\underset{(0)}{\check{\mathbf{e}}}\hat{\#}1 \tag{57}$$

$$\hat{\boldsymbol{d}}\ \hat{\#}\underset{(1)}{\check{\mathbf{e}}} = -i\,k\,\underset{(0)}{\check{\mathbf{e}}}\hat{\#}1 \tag{58}$$

$$\hat{\boldsymbol{d}}\ \hat{\#}\underset{(1)}{\check{\mathbf{h}}} = -i\,k\,\underset{(0)}{\check{\mathbf{h}}}\hat{\#}1 \tag{59}$$

$$\hat{\boldsymbol{d}}\ \underset{(0)}{\check{\mathbf{e}}} - i\,k\,\underset{(1)}{\check{\mathbf{e}}} = i\omega\mu\hat{\#}\underset{(1)}{\check{\mathbf{h}}} \tag{60}$$

$$\hat{\boldsymbol{d}}\ \underset{(0)}{\check{\mathbf{h}}} - i\,k\,\underset{(1)}{\check{\mathbf{h}}} = -i\omega\epsilon\hat{\#}\underset{(1)}{\check{\mathbf{e}}} \tag{61}$$

It follows from (60) and (61) that provided $k^2 - \omega^2\epsilon\mu \neq 0$ then

$$(k^2 - \omega^2\epsilon\mu)\,\hat{\#}\underset{(1)}{\check{\mathbf{h}}} = -i\,k\hat{\#}\hat{\boldsymbol{d}}\ \underset{(0)}{\check{\mathbf{h}}} + i\,\omega\,\epsilon\hat{\boldsymbol{d}}\ \underset{(0)}{\check{\mathbf{e}}} \tag{62}$$

$$(k^2 - \omega^2\epsilon\mu) \underset{(1)}{\check{\mathbf{e}}} = i\,\omega\mu\hat{\#}\hat{\boldsymbol{d}}\ \underset{(0)}{\check{\mathbf{h}}} - i\,k\,\hat{\boldsymbol{d}}\ \underset{(0)}{\check{\mathbf{e}}} \tag{63}$$

Thus the *transverse* forms $\underset{(1)}{\check{\mathbf{e}}}$ and $\underset{(1)}{\check{\mathbf{h}}}$ are specified in terms of the derivatives of the *longitudinal* forms $\underset{(0)}{\check{\mathbf{e}}}$ and $\underset{(0)}{\check{\mathbf{h}}}$.

If one applies $\hat{\boldsymbol{d}}\ \hat{\#}$ to (63) and uses (58) one finds that $\underset{(0)}{\check{\mathbf{e}}}$ must be an eigen-form of the transverse cylinder Laplacian:

$$\hat{\boldsymbol{d}}\ \hat{\#}\hat{\boldsymbol{d}}\ \underset{(0)}{\check{\mathbf{e}}} = (k^2 - \omega^2\epsilon\mu)\ \underset{(0)}{\check{\mathbf{e}}}\ \hat{\#}1 \tag{64}$$

Similarly if one applies $\hat{\boldsymbol{d}}$ to (62) and uses (59) one finds that $\underset{(0)}{\check{\mathbf{h}}}$ must also be an eigen-form of the transverse cylinder Laplacian:

$$\hat{\boldsymbol{d}}\ \hat{\#}\hat{\boldsymbol{d}}\ \underset{(0)}{\check{\mathbf{h}}} = (k^2 - \omega^2\epsilon\mu)\ \underset{(0)}{\check{\mathbf{h}}}\ \hat{\#}1 \tag{65}$$

But the geometry of the domain $\mathcal{D}$ (given here by z = constant, $0 \le r \le a, 0 < \phi \le 2\pi$) determines these eigen-forms once the boundary conditions are specified. For a perfectly conducting cylinder one must choose appropriate eigen-forms for these longitudinal fields so that the proper boundary conditions for $\underset{(1)}{\mathbf{e}}$ and $\underset{(1)}{\mathbf{h}}$ are satisfied at $r = a$. One requires that $\underset{(0)}{\check{\mathbf{e}}}$ be a Dirichlet mode (Φ_N) and $\underset{(0)}{\check{\mathbf{h}}}$ be a Neumann mode (Ψ_M). The eigenvalues of these modes follow by solving (41) and (46). Thus the parameters k and ω are constrained to (k_N, ω_N) satisfying

$$k_N^2 - \epsilon\mu\omega_N^2 = -\beta_N^2 \tag{66}$$

or (k_M, ω_M) satisfying or

$$k_M^2 - \epsilon\mu\omega_M^2 = -\alpha_M^2 \tag{67}$$

Since the eigenvalues α_M^2, β_N^2 are real one sees that *propagating* modes correspond to real k roots of this equation. Configurations in which the roots k are pure imaginary are called *evanescent* since they attenuate as a function of z.

It is now possible to verify that field configurations in the hollow cylinder satisfying all the above Maxwell equations decouple into two sets. Those with $\underset{(0)}{\check{\mathbf{h}}} = 0$, $\underset{(0)}{\check{\mathbf{e}}} = \Phi_N$ and $k_N^2 - \epsilon\mu\omega_N^2 = -\beta_N^2$ are termed cylindrical TM modes. It follows from (62) and (63) that for these modes

$$(k^2 - \omega^2\epsilon\mu)\,\hat{\#}\underset{(1)}{\check{\mathbf{h}}} = i\,\omega\,\epsilon\hat{\boldsymbol{d}}\;\underset{(0)}{\check{\mathbf{e}}} \tag{68}$$

$$(k^2 - \omega^2\epsilon\mu)\;\underset{(1)}{\check{\mathbf{e}}} = -i\,k\,\hat{\boldsymbol{d}}\;\underset{(0)}{\check{\mathbf{e}}} \tag{69}$$

Those with $\underset{(0)}{\check{\mathbf{e}}} = 0$, $\underset{(0)}{\check{\mathbf{h}}} = \Psi_M$ and $k_M^2 - \epsilon\mu\omega_M^2 = -\alpha_M^2$ are termed cylindrical TE modes. It follows from (62) and (63) that for these modes

$$(k^2 - \omega^2\epsilon\mu)\,\hat{\#}\underset{(1)}{\check{\mathbf{h}}} = -i\,k\hat{\#}\hat{\boldsymbol{d}}\;\underset{(0)}{\check{\mathbf{h}}} \tag{70}$$

$$(k^2 - \omega^2\epsilon\mu)\;\underset{(1)}{\check{\mathbf{e}}} = i\,\omega\mu\hat{\#}\hat{\boldsymbol{d}}\;\underset{(0)}{\check{\mathbf{h}}} \tag{71}$$

For each TE mode configuration all propagating fields characterised by the mode label N have a wavelength $\frac{2\pi}{k_N} = \frac{2\pi}{\sqrt{\epsilon\mu\omega_N^2 - \beta_N^2}}$, a phase speed $\omega_N/k_N = \frac{\omega_N}{\sqrt{\epsilon\mu\omega_N^2 - \beta_N^2}}$ and a group speed $(\frac{d\,\omega}{d\,k})_N = \frac{1}{\epsilon\mu}\frac{k_N}{\omega_N}$ determined by the geometry of $\mathcal{D}$. Similarly each propagating TM mode configuration characterised by the mode label M has a wavelength $\frac{2\pi}{k_M} = \frac{2\pi}{\sqrt{\epsilon\mu\omega_M^2 - \alpha_M^2}}$, a phase speed $\omega_M/k_M = \frac{\omega_M}{\sqrt{\epsilon\mu\omega_M^2 - \alpha_M^2}}$ and a group speed $(\frac{d\,\omega}{d\,k})_M = \frac{1}{\epsilon\mu}\frac{k_M}{\omega_M}$. Since the source free system is linear in all fields, more general configurations can be generated by superposition of all possible TE and TM modes. Since the cylinder is hollow the most general field configuration follows by adding any static electric and/or magnetic field to these that are compatible with the boundary conditions [6].

[6] If the cylinder has a metallic concentric cylindrical core there also exist propagating modes with both $\underset{(0)}{\hat{\mathbf{h}}}$ and $\underset{(0)}{\hat{\mathbf{e}}}$ simultaneously zero. Such TEM modes cannot arise in the hollow cylinder under consideration here

For completeness one may explicitly solve (41) and (46) to find the transverse cylinder mode structure. It is convenient as usual [7] to complexify. Using regularity at $r = 0$ and periodicity in ϕ one has the TE mode labels $N \equiv (m, p)$ with $m = 0, \pm 1, \pm 2, \dots$. Let x_{mp} denote the p-th root of the J_m Bessel function, i.e $J_m(x_{mp}) = 0$. Then $\beta_N \equiv \beta_{mp} = \frac{x_{mp}}{a}$ and

$$\Phi_N(r, \phi) = \mathcal{N}_{mp}\, J_m(x_{mp}\frac{r}{a})e^{im\phi} \tag{72}$$

Similarly one has the TM mode labels $M \equiv (m, p)$ with $m = 0, \pm 1, \pm 2, \dots$. Let x'_{mp} denote the p-th root of the equation $J'_m(x'_{mp}) = 0$. Then $\alpha_M \equiv \alpha_{mp} = \frac{x'_{mp}}{a}$ and

$$\Psi_M(r, \phi) = \mathcal{M}_{mp}\, J_m(x'_{mp}\frac{r}{a})e^{im\phi} \tag{73}$$

In a medium satisfying Ohm's law $\underset{(2)}{J} = \sigma \underset{(2)}{\mathbf{E}}$ with constant scalar conductivity σ, the dispersion relations above are modified. For any field mode m, p one replaces the factor e^{ikz} for real k by $e^{-\Gamma z}$ for complex $\Gamma = \alpha + ik$ with real attenuation parameter $\alpha > 0$. Then the dispersion relation becomes:

$$\Gamma^2 - \Gamma_0^2 = x_{mp}^2$$

where $\Gamma_0^2 = i\mu\omega\sigma - \omega^2\epsilon\mu$. Thus the real parameters k, α that determine the propagation characteristics of a particular mode are determined by the real and imaginary parts of the above complex dispersion relation:

$$x_{mp}^2 = \alpha^2 - k^2 + \omega^2\epsilon\mu$$

$$2\alpha k - \mu\omega\sigma = 0$$

For positive $\alpha = \frac{\mu\omega\sigma}{2k}$ the loci with real ω and $k \neq 0$ satisfy:

$$x_{mp}^2 = \frac{\mu^2\omega^2\sigma^2}{4k^2} - k^2 + \omega^2\epsilon\mu$$

[7] One then uses mode orthonormality relations $\int_{\mathcal{D}} \bar{\Phi}_M\, \Phi_N\, \hat{\#}1 = \mathcal{N}_N^2\, \delta_{NM}$, $\int_{\mathcal{D}} \bar{\Psi}_M\, \Psi_N\, \hat{\#}1 = \mathcal{M}_N^2\, \delta_{NM}$

and determine the manner in which the sharp cut-off for $\sigma = 0$ is modified when $\sigma \neq 0$.

The above cylindrical coframe is also readily applicable to the theory of the sector horn. A sector horn with a rectangular cross-section is composed of the union of four planes in space. In cylindrical polar coordinates they are given as

$$A: \quad \phi = 0 \tag{74}$$

$$C: \quad \phi = \phi_0 \tag{75}$$

$$B: \quad z = 0 \tag{76}$$

$$D: \quad z = z_0 \tag{77}$$

Using Dirichlet and Neumann modes associated with the domain $\mathcal{D}$ given by any surface with $r = \textit{constant}$ and inside the surface $S = A \cup B \cup C \cup D$. A global analysis of the pure mode structure of the horn satisfying perfectly conducting boundary conditions on S can be made in terms of its spectral content.

9 Cylinder modes excited by internal sources

In practice the source free pure cylinder modes above are excited by external agencies. Some technological skill is often required to excite pure TE or TM propagating modes. Mathematically these require specification of precise initial conditions. When sources j, with mobile charge exist in the cylinder they act as forcing terms and may excite a superposition of allowable modes. Guided by the above mode analysis we explore modes of the form

$$\underset{(0)}{\hat{\mathbf{h}}} = 0 \tag{78}$$

$$\underset{(1)}{\hat{\mathbf{h}}}(t, z, r, \phi) = \sum_N I_N^E(t, z)\, \hat{\#}\boldsymbol{d}\, \Phi_N(r, \phi) \tag{79}$$

$$\underset{(0)}{\hat{\mathbf{e}}}(t, z, r, \phi) = \sum_N \gamma_N^E(t, z)\, \Phi_N(r, \phi) \tag{80}$$

$$\underset{(1)}{\hat{\mathbf{e}}}(t,z,r,\phi) = \sum_N V_N^E(t,z)\,\boldsymbol{d}\,\Phi_N(r,\phi) \tag{81}$$

where the functions I_N^E, γ_N^E, V_N^E must be determined.

It follows immediately that such modes satisfy

$$\hat{\boldsymbol{d}}\;\underset{(1)}{\hat{\mathbf{e}}} = 0 \tag{82}$$

and

$$\hat{\boldsymbol{d}}\;\underset{(1)}{\hat{\mathbf{h}}} = -\sum_N \beta_N^2\, I_N^E \Phi_N \,\hat{\#}1 \tag{83}$$

Inserting the above expressions into the Maxwell system with sources (29), (30), (32), (33), (36), (37) and integrating over $\mathcal{D}$ using the orthogonality relations for the Φ_N modes yields the E-mode reduced system:

$$\beta_M^2\, I_M^E(t,z) + \epsilon\,\dot{\gamma}_M^E(t,z) = -\frac{1}{{\mathcal{N}_M}^2}\int_{\mathcal{D}} \underset{(0)}{\hat{J}}(t,z,r,\phi)\,\Phi_M(r,\phi)\,\hat{\#}1 \tag{84}$$

$$I_M^{E\prime}(t,z) + \epsilon\,\dot{V}_M^E = -\frac{1}{\mathcal{N}_M\beta_M^2}\int_{\mathcal{D}} \boldsymbol{d}\,\Phi_M(r,\phi)\wedge \underset{(1)}{\hat{J}}(t,z,r,\phi) \tag{85}$$

$$V_N^{E\prime}(t,z) - \gamma_N^E(t,z) + \mu\,\dot{I}_N^E(t,z) = 0 \tag{86}$$

$$V_M^E(t,z)\beta_N^2 - \gamma_M^{E\prime}(t,z) = -\frac{1}{\epsilon\,{\mathcal{N}_M}^2}\int_{\mathcal{D}} \underset{(0)}{\rho}\,(t,z,r,\phi)\,\Phi_M(r,\phi)\,\hat{\#}1 \tag{87}$$

where $f' \equiv \frac{\partial}{\partial z} f$ for any scalar field f. In these equations the sources are constrained to satisfy the conservation relation (38) that here takes the form:

$$\hat{\boldsymbol{d}}\;\underset{(1)}{\hat{J}} + \left(\frac{\partial}{\partial z}\underset{(0)}{\hat{J}} + \frac{\partial}{\partial t}\underset{(0)}{\rho}\right)\hat{\#}1 = 0 \tag{88}$$

Since

$$\hat{\boldsymbol{d}}\,(\Phi_M \wedge \underset{(1)}{\hat{J}}) = \hat{\boldsymbol{d}}\;\Phi_M \wedge \underset{(1)}{\hat{J}} + \Phi_M \hat{\boldsymbol{d}}\;\underset{(1)}{\hat{J}}$$

and $\Phi_M|_{\partial\mathcal{D}} = 0$ one finds from (88) that

$$\int_{\mathcal{D}} \boldsymbol{d}\,\Phi_M \wedge \underset{(1)}{\hat{J}} = \int_{\mathcal{D}} \Phi_M(r,\phi)\,(\underset{(0)}{\hat{J}}{}'(t,z,r,\phi) + \underset{(0)}{\dot{\rho}}(t,z,r,\phi))\,\hat{\#}1 \qquad (89)$$

Suppose any 0−form $\mathcal{F}$ depends on r,ϕ and a collection of other variables and one has:

$$\mathcal{F}(r,\phi,\ldots) = \sum_N <\mathcal{F}>_N (\ldots)\Phi_N(r,\phi) \qquad (90)$$

then by orthogonality of the Φ_N:

$$<\mathcal{F}>_N (\ldots) = \frac{1}{\mathcal{N}_N{}^2}\int_{\mathcal{D}} \mathcal{F}(r,\phi,\ldots)\,\Phi_N\,\hat{\#}1 \qquad (91)$$

In terms of the projected sources $<\underset{(0)}{\hat{J}}>_N (t,z)$ and $<\underset{(0)}{\rho}>_N (t,z)$ the reduced E-mode system

$$\dot{\gamma}^E_M + \frac{\beta_M^2}{\epsilon}\,I^E_M = -\frac{1}{\epsilon} <\underset{(0)}{\hat{J}}>_M \qquad (92)$$

$$\dot{V}^E_M + \frac{1}{\epsilon}\,I^{E\prime}_M = -\frac{1}{\epsilon\,\beta_M^2}\left(\frac{\partial}{\partial z} <\underset{(0)}{\hat{J}}>_M + \frac{\partial}{\partial t} <\underset{(0)}{\rho}>_M\right) \qquad (93)$$

$$\gamma^E_M = V^{E\prime}_M + \mu\,\dot{I}^E_M \qquad (94)$$

$$\gamma^{E\prime}_M - \beta_M^2\,V^E_M = \frac{1}{\epsilon} <\underset{(0)}{\rho}>_M \qquad (95)$$

is seen to be system of p.d.e.'s for $I^E_M(t,z)$, $V^E_M(t,z)$, $\gamma^E_M(t,z)$.

In a similar manner we expect to find a mode system generated from expansions with

$$\underset{(0)}{\hat{\mathbf{e}}} = 0 \qquad (96)$$

$$\underset{(1)}{\hat{\mathbf{e}}}(t,z,r,\phi) = \sum_N V^H_N(t,z)\,\hat{\#}\boldsymbol{d}\,\Psi_N(r,\phi) \qquad (97)$$

$$\underset{(0)}{\hat{\mathbf{h}}}(t,z,r,\phi) = \sum_N \gamma_N^H(t,z)\,\Psi_N(r,\phi) \tag{98}$$

$$\underset{(1)}{\hat{\mathbf{h}}}(t,z,r,\phi) = \sum_N I_N^H(t,z)\,\boldsymbol{d}\,\Psi_N(r,\phi) \tag{99}$$

It follows immediately that for these modes

$$\hat{\boldsymbol{d}}\;\underset{(1)}{\hat{\mathbf{h}}} = 0 \tag{100}$$

and

$$\hat{\boldsymbol{d}}\;\underset{(1)}{\hat{\mathbf{e}}} = -\sum_N V_N^H \alpha_N^2\,\Psi_N\,\hat{\#}1 \tag{101}$$

Inserting the above expansions in the Maxwell system and using the Ψ_M orthogonality to project as before yields

$$\underset{(0)}{\hat{J}} = 0 \tag{102}$$

$$\epsilon\,\dot{V}_M^H + \gamma_M^H - I_M^{H\prime} = \frac{1}{\alpha_M^2\,\mathcal{M}_M^2}\int_{\mathcal{D}} \boldsymbol{d}\,\Psi_M \wedge \hat{\#}\,\underset{(1)}{\hat{J}} \tag{103}$$

$$\mu\,\dot{\gamma}_M^H - V_M^H\,\alpha_M^2 = 0 \tag{104}$$

$$\mu\,\dot{I}_M^H - V_M^{H\prime} = 0 \tag{105}$$

$$\gamma_M^{H\prime} - I_M^H\,\alpha_M^2 = 0 \tag{106}$$

$$\underset{(0)}{\rho} = 0 \tag{107}$$

Clearly such configurations are only excited by particular types of source. Using Stokes theorem on the identity

$$\int_{\mathcal{D}} \hat{\boldsymbol{d}}\;\Psi_M \wedge \hat{\#}\,\underset{(1)}{\hat{J}} = \int_{\mathcal{D}} \hat{\boldsymbol{d}}\;(\Psi_M \hat{\#}\,\underset{(1)}{\hat{J}}) - \int_{\mathcal{D}} \Psi_M \hat{\boldsymbol{d}}\;\hat{\#}\,\underset{(1)}{\hat{J}} \tag{108}$$

gives

$$\int_{\mathcal{D}} \hat{\boldsymbol{d}}\,\Psi_M \wedge \hat{\#} \underset{(1)}{\hat{J}} = \int_{\partial\mathcal{D}} \Psi_M \hat{\#} \underset{(1)}{\hat{J}} - \int_{\mathcal{D}} \Psi_M \hat{\boldsymbol{d}}\,\hat{\#} \underset{(1)}{\hat{J}} \tag{109}$$

Since Ψ_M does not vanish on $\partial\mathcal{D}$ one sees that this system can get excitations from boundary currents and (103) can be written

$$\epsilon\,\dot{V}^H_M + \gamma^H_M - I^{H\prime}_M + \frac{1}{\alpha^2_M} << \hat{\#}\,\hat{\boldsymbol{d}}\,\hat{\#} \underset{(1)}{\hat{J}} >>_M - \frac{1}{\alpha^2_M\,\mathcal{M}^2_M} \int_{\partial\mathcal{D}} \Psi_M \hat{\#} \underset{(1)}{\hat{J}} = 0 \tag{110}$$

where here,

$$<< \mathcal{F} >>_M \equiv \frac{1}{{\mathcal{M}_M}^2} \int_{\mathcal{D}} \mathcal{F}\Psi_M \hat{\#} 1$$

A general mixed configuration in a hollow cylinder with arbitrary internal sources, $\underset{(0)}{\rho}(z,t,r,\phi)$, $\underset{(0)}{\hat{J}}(z,t,r,\phi)$, $\underset{(1)}{\hat{J}}(z,t,r,\phi)$ satisfying (38), will take the form

$$\underset{(1)}{\hat{\mathbf{e}}} = \sum_N V^E_N\,\boldsymbol{d}\,\Phi_N + \sum_M V^H_M \hat{\#} \boldsymbol{d}\,\Psi_M \tag{111}$$

$$\underset{(1)}{\hat{\mathbf{h}}} = \sum_N I^E_N\,\hat{\#} \boldsymbol{d}\,\Phi_N + \sum_M I^H_M \boldsymbol{d}\,\Psi_M \tag{112}$$

$$\underset{(0)}{\hat{\mathbf{e}}} = \sum_N \gamma^E_N\,\Phi_N \tag{113}$$

$$\underset{(0)}{\hat{\mathbf{h}}} = \sum_M \gamma^H_M \Psi_M \tag{114}$$

and the p.d.e.'s for $V^E_N, V^H_M, I^E_N, I^H_M, \gamma^E_N, \gamma^H_M$ derived from (29), (30), (32),(33), (36), (37) will be fully coupled[8] in general.

If additionally one has inside the cylinder fields due to sources external to $\mathcal{D}$ then they must be solutions to the above system with $\underset{(0)}{\rho}(z,t,r,\phi) =$

[8]Note that when all the sources vanish the equations can be decoupled into TE and TM cylindrical modes with the various amplitudes $V_,\ I_,\ \gamma_,$ all proportional to $\exp(k\,z - \omega\,t)$ with parameters k and ω constrained by the pure mode dispersion relations above.

0, $\underset{(0)}{\hat{J}}(z,t,r,\phi)=0$, $\underset{(1)}{\hat{J}}(z,t,r,\phi)=0$ satisfying appropriate boundary conditions. These will arise if one attempts to control dynamic sources with external fields produced by prescribed (control) charges or currents. Thus, for example, one has the exact static solutions

$$\underset{(1)}{\mathbf{e}}^{ext}=\boldsymbol{d}\,\mathcal{V}(z,r,\phi)$$

$$\underset{(1)}{\mathbf{h}}^{ext}=\boldsymbol{d}\,\mathcal{M}(z,r,\phi)$$

for any potentials satisfying

$$\boldsymbol{d}\,\#\boldsymbol{d}\,\mathcal{V}=0$$
$$\boldsymbol{d}\,\#\boldsymbol{d}\,\mathcal{M}=0$$

in $\mathcal{D}$ and satisfying metallic boundary conditions.

10 The initial value problem

If the compatible sources in the systems above are prescribed one may formulate an initial value problem for the resulting fields. This will be illustrated for the reduced E-mode system. From this it is straightforward to deduce that $\gamma^E_M(t,z)$ satisfies

$$\ddot{\gamma}^E_M-v^2\gamma^{E\,\prime\prime}_M+v^2\beta^2_M\gamma^E_M=\mathcal{J}^E_M \tag{115}$$

where $\mathcal{J}^E_M=-\frac{1}{\epsilon}<\underset{(0)}{\dot{\hat{J}}}>_M-\frac{v^2}{\epsilon}<\underset{(0)}{\rho}{}'>_M$ and $v^2=\frac{1}{\epsilon\mu}$.

The causal solution (having $\gamma^E_M=0$ for $t<0$) of this partial differential equation with prescribed values of $\gamma^E_M(0,z)$ and $\dot{\gamma}^E_M(0,z)$ has been exhaustively studied in the literature, see e.g. [6]. If the data and sources are sufficiently smooth one has:

$$\begin{aligned}\gamma_M^E(t,z) &= \frac{1}{2}\{\gamma_M^E(0,z-vt)+\gamma_M^E(0,z+vt)\}\\ &+\frac{1}{2v}\int_{z-vt}^{z+vt}\dot{\gamma}_M^E(0,\zeta)\,J_0(\beta_M^2\sqrt{v^2t^2-(z-\zeta)^2})\,d\zeta\\ &-\frac{vt\beta_M}{2}\int_{z-vt}^{z+vt}\gamma_M^E(0,\zeta)\frac{J_1(\beta_M^2\sqrt{v^2t^2-(z-\zeta)^2})}{\sqrt{v^2t^2-(z-\zeta)^2}}\,d\zeta\\ &+\frac{1}{2v}\int_0^t\int_{z-v(t-t')}^{z+v(t-t')}\mathcal{J}_M^E(t',\zeta)\,J_0(\beta_M^2\sqrt{v^2(t-t')^2-(z-\zeta)^2})\,dt'\,d\zeta\end{aligned} \tag{116}$$

This solution can be generalized to a distributional solution for sources modeled by moving point charges. The remaining fields can be computed in terms of γ_M^E.

11 The Maxwell system inside a conical guide

Suppose a hollow perfectly conducting cone with apex angle $2\theta_0$ is given in spherical polar coordinates with $\{\xi^1=\theta,\xi^2=\phi,\xi^3=r\}$ by $\theta=\theta_0$: A convenient coframe in these coordinates is

$$\{e^1=r\,\boldsymbol{d}\,\theta,\quad e^2=r\,\sin\theta\,\boldsymbol{d}\,\phi,\quad e^3=\boldsymbol{d}\,r\}$$

with the tensor $\hat{g}=e^1\otimes e^1+e^2\otimes e^2=r^2(\boldsymbol{d}\,\theta\otimes\boldsymbol{d}\,\theta+\sin^2\theta\boldsymbol{d}\,\phi\otimes\boldsymbol{d}\,\phi)$. One now has $\mathcal{L}_{\frac{\partial}{\partial r}}\hat{g}=\frac{2}{r}\hat{g}$ and hence $\mathcal{L}_{\frac{\partial}{\partial r}}\hat{\#}1=\frac{2}{r}\hat{\#}1$. This coframe has the structure equations:

$$\boldsymbol{d}\,e^1=\frac{1}{r}\,e^3\wedge e^1$$

$$\boldsymbol{d}\,e^2=\frac{1}{r}\,e^3\wedge e^2+\frac{\cot\theta}{r}\,e^1\wedge e^2$$

$$\boldsymbol{d}\,e^3=0$$

Using the 2+1 decompositions

$$\underset{(1)}{\mathbf{e}}=\underset{(0)}{\hat{\mathbf{e}}}\wedge\boldsymbol{d}\,r+\underset{(1)}{\hat{\mathbf{e}}} \tag{117}$$

$$\underset{(1)}{\mathbf{h}} = \underset{(0)}{\hat{\mathbf{h}}} \wedge \boldsymbol{d}\, r + \underset{(1)}{\hat{\mathbf{h}}} \tag{118}$$

the Maxwell equation $\boldsymbol{d}\,\underset{(1)}{\mathbf{e}} = -\mu \underset{(2)}{\dot{\mathbf{H}}}$ yields:

$$\hat{\boldsymbol{d}}\,\underset{(0)}{\hat{\mathbf{e}}} \wedge \boldsymbol{d}\, r + \hat{\boldsymbol{d}}\,\underset{(1)}{\hat{\mathbf{e}}} + \boldsymbol{d}\, r \wedge \mathcal{L}_{\frac{\partial}{\partial r}}\underset{(1)}{\hat{\mathbf{e}}} = -\mu \hat{\#}\underset{(0)}{\dot{\hat{\mathbf{h}}}} - \mu \hat{\#}\underset{(1)}{\dot{\hat{\mathbf{h}}}} \wedge \boldsymbol{d}\, r$$

or the pair

$$\hat{\boldsymbol{d}}\,\underset{(0)}{\hat{\mathbf{e}}} - \mathcal{L}_{\frac{\partial}{\partial r}}\underset{(1)}{\hat{\mathbf{e}}} = -\mu \hat{\#}\underset{(1)}{\dot{\hat{\mathbf{h}}}} \tag{119}$$

$$\hat{\boldsymbol{d}}\,\underset{(1)}{\hat{\mathbf{e}}} = -\mu \hat{\#}\underset{(0)}{\dot{\hat{\mathbf{h}}}} \tag{120}$$

Similarly with the 2+1 split

$$\underset{(2)}{J} = \underset{(1)}{\hat{J}} \wedge \boldsymbol{d}\, r + \underset{(2)}{\hat{J}} \tag{121}$$

the Maxwell equation $\boldsymbol{d}\,\underset{(1)}{\mathbf{h}} = \epsilon \underset{(2)}{\dot{\mathbf{E}}} + \underset{(2)}{J}$ yields the pair

$$\hat{\boldsymbol{d}}\,\underset{(0)}{\hat{\mathbf{h}}} - \mathcal{L}_{\frac{\partial}{\partial r}}\underset{(1)}{\hat{\mathbf{h}}} = \epsilon \hat{\#}\underset{(1)}{\dot{\hat{\mathbf{e}}}} + \underset{(1)}{\hat{J}} \tag{122}$$

$$\hat{\boldsymbol{d}}\,\underset{(1)}{\hat{\mathbf{h}}} = \epsilon \hat{\#}\underset{(0)}{\dot{\hat{\mathbf{e}}}} + \underset{(2)}{\hat{J}} \tag{123}$$

The structure of the Maxwell system for the cone so far coincides with that for the cylinder. However this changes when one splits the remaining Maxwell equations since $\hat{\#}1$ is not invariant under Lie differentiation with respect to radial variations (generated by $\frac{\partial}{\partial r}$). Thus with the 2+1 splits

$$\underset{(2)}{\mathbf{E}} = \hat{\#}\underset{(0)}{\hat{\mathbf{e}}} + (\hat{\#}\underset{(1)}{\hat{\mathbf{e}}}) \wedge \boldsymbol{d}\, r \tag{124}$$

and

$$\boldsymbol{d}\,\underset{(2)}{\mathbf{E}} = \boldsymbol{d}\, r \wedge \big(\mathcal{L}_{\frac{\partial}{\partial r}}\big(\underset{(0)}{\hat{\mathbf{e}}}\,\hat{\#}1\big) + \hat{\boldsymbol{d}}\,\hat{\#}\underset{(1)}{\hat{\mathbf{e}}}\big) \tag{125}$$

the Maxwell equation $\epsilon \boldsymbol{d} \underset{(2)}{\mathbf{E}} = \underset{(0)}{\rho} \# 1$ yields:

$$\hat{\boldsymbol{d}} \,\hat{\#} \underset{(1)}{\hat{\mathbf{e}}} + (\mathcal{L}_{\frac{\partial}{\partial r}} \underset{(0)}{\hat{\mathbf{e}}} + \frac{2}{r} \underset{(0)}{\hat{\mathbf{e}}}) \,\hat{\#} 1 = \frac{1}{\epsilon} \underset{(0)}{\rho} \,\hat{\#} 1 \tag{126}$$

Similarly the final Maxwell equation $\boldsymbol{d} \underset{(2)}{\mathbf{H}} = 0$ yields

$$\hat{\boldsymbol{d}} \,\hat{\#} \underset{(1)}{\hat{\mathbf{h}}} + (\mathcal{L}_{\frac{\partial}{\partial r}} \underset{(0)}{\hat{\mathbf{h}}} + \frac{2}{r} \underset{(0)}{\hat{\mathbf{h}}}) \,\hat{\#} 1 = 0 \tag{127}$$

Finally 4-current conservation yields in this case:

$$\hat{\boldsymbol{d}} \underset{(1)}{\hat{J}} + \mathcal{L}_{\frac{\partial}{\partial r}} \underset{(2)}{\hat{J}} + \underset{(0)}{\dot{\rho}} \,\hat{\#} 1 = 0 \tag{128}$$

12 Source free pure harmonic conical modes

The analysis of the pure cylinder harmonic modes emphasised the role played by eigen-forms of the Laplacian associated with the coordinate surface $\mathcal{D}$ given by $z = constant$ in cylindrical polar coordinates. Due to translational symmetry the geometry of this cross-section of the cylinder was constant. For the cone the analagous surface is given in spherical polar coordinates (centred at the cone apex) by $r = constant$. Although for different constants such surfaces are similar they do not share the translation symmetry property exhibited by the cylinder. However one may explore the consequences of using the eigen-forms of the $2-$sphere to explore the pure electromagnetic mode structure of the cone. The scalar eigen-forms for a spherical patch $\mathcal{D}$ will be denoted Y_l^m and for some numbers m, l they satisfy:

$$\boldsymbol{d} \,\hat{\#} \boldsymbol{d} Y_l^m = -\frac{1}{r^2} l(l+1) Y_l^m \hat{\#} 1 \tag{129}$$

For the complete sphere $l = 0, 1, 2, \ldots$ and for each l, the range $m = -l, -l+1, \ldots l$ ensures that Y_l^m is a polynomial in $\cos\theta$ and periodic in ϕ.

However for the cone under consideration the domain $\mathcal{D}$ does not cover the surface of a full sphere so the range of m and l are not in general those appropriate for the full sphere. If one defines $\mathcal{D}$ as $r = r_0$ and parametrises it by θ, ϕ with $0 \leq \phi < 2\pi,\ 0 \leq \theta \leq \theta_0$ then periodicity of all fields in ϕ is demanded so m must be integer. The value of l will be fixed by demanding that the electromagnetic fields satisfy the perfectly conducting boundary conditions on $\partial\mathcal{D}$ given by $\theta = \theta_0$. The eigen-forms $Y_l^m(\theta,\phi)$ are then to be expressed as $e^{im\phi}P_l^m(\cos\theta)$ in terms of associated Legendre functions $P_l^m(\mu)$.

From applying $\hat{\boldsymbol{d}}\ \hat{\#}$ to (119) and using (126) and (123) one finds that $\underset{(0)}{\hat{\mathbf{e}}}$ must satisfy:

$$\hat{\boldsymbol{d}}\ \hat{\#}\hat{\boldsymbol{d}}\ \underset{(0)}{\hat{\mathbf{e}}} + \left((\underset{(0)}{\hat{\mathbf{e}}}{}' + \frac{2}{r}\underset{(0)}{\hat{\mathbf{e}}})\hat{\#}1 \right)' - \mu\epsilon\,\underset{(0)}{\ddot{\hat{\mathbf{e}}}}\,\hat{\#}1 = 0 \tag{130}$$

Similarly applying $\hat{\boldsymbol{d}}\ \hat{\#}$ to (122) and using (127) and (120) one finds that $\underset{(0)}{\hat{\mathbf{h}}}$ must satisfy:

$$\hat{\boldsymbol{d}}\ \hat{\#}\hat{\boldsymbol{d}}\ \underset{(0)}{\hat{\mathbf{h}}} + \left((\underset{(0)}{\hat{\mathbf{h}}}{}' + \frac{2}{r}\underset{(0)}{\hat{\mathbf{h}}})\hat{\#}1 \right)' - \mu\epsilon\,\underset{(0)}{\ddot{\hat{\mathbf{h}}}}\,\hat{\#}1 = 0 \tag{131}$$

where now, for any form $\mathcal{F}$, $\mathcal{F}' \equiv \mathcal{L}_{\frac{\partial}{\partial r}}\mathcal{F}$.

Guided by the separability structure of the cylindrical pure modes one now expects to solve (130) and (131) with the forms:

$$\underset{(0)}{\hat{\mathbf{e}}}(t,r,\theta,\phi) = C_e \exp(i\omega t)\Phi_e(r)\Phi_N(\theta,\phi) \tag{132}$$

$$\underset{(0)}{\hat{\mathbf{h}}}(t,r,\theta,\phi) = C_h \exp(i\omega t)\Phi_h(r)\Psi_M(\theta,\phi) \tag{133}$$

for some constants ω, C_e, C_h. The Dirichlet modes are

$$\Phi_N(\theta,\phi) = Y_l^m(\theta,\phi) \tag{134}$$

while the Neumann modes are

$$\Psi_N(\theta,\phi) = Y_L^m(\theta,\phi) \tag{135}$$

with $N \equiv (m,l)$, $M \equiv (m,L)$ for some m,l,L to be determined. Using (129) to determine Φ_e and Φ_h, one finds that instead of the functions $\exp(ikr)$ defining plane fronted propagating waves for some k as in the pure cylindrical mode case one must have:

$$\Phi_e(r) = \frac{\mathcal{Z}_l\left(\frac{\omega}{v}r\right)}{r} \tag{136}$$

$$\Phi_h(r) = \frac{\mathcal{Z}_L(\frac{\omega}{v}r)}{r} \tag{137}$$

defining spherical fronted propagating waves for some M and N. The function $\mathcal{Z}_l$ is any solution of the spherical Bessel equation:

$$\zeta^2 \frac{d^2}{d\zeta^2}\mathcal{Z}_l(\zeta) + 2\zeta\frac{d}{d\zeta}\mathcal{Z}_l(\zeta) + \big(\zeta^2 - l(l+1)\big)\mathcal{Z}_l(\zeta) = 0 \tag{138}$$

A basis of such solutions is the set containing the real functions often denoted $j_l(\zeta)$ and $y_l(\zeta)$.[9] Outgoing (ingoing) complex propagating fields can be constructing in terms of the spherical Hankel functions:

$$h_l^{1,2} = j_l \pm i\, y_l \tag{139}$$

These are really only appropriate for a truncated cone (frustum). If the apex of the cone is maintained then the y_l functions must be excluded since they are singular at $r = 0$.

Furthermore from (119) and (122) one finds

$$\underset{(1)}{\hat{\mathbf{h}}}{}'' + \frac{\omega^2}{v^2}\underset{(1)}{\hat{\mathbf{h}}} = (\hat{\boldsymbol{d}}\ \underset{(0)}{\hat{\mathbf{h}}})' - i\omega\,\epsilon\hat{\#}\boldsymbol{d}\,\underset{(0)}{\hat{\mathbf{e}}} \tag{140}$$

$$\underset{(1)}{\hat{\mathbf{e}}}{}'' + \frac{\omega^2}{v^2}\underset{(1)}{\hat{\mathbf{e}}} = (\hat{\boldsymbol{d}}\ \underset{(0)}{\hat{\mathbf{e}}})' + i\omega\,\mu\hat{\#}\boldsymbol{d}\,\underset{(0)}{\hat{\mathbf{h}}} \tag{141}$$

[9] In terms of the Bessel functions J_l and Y_l one has $j_l(\zeta) = \sqrt{\frac{\pi}{2\zeta}}\,J_{l+\frac{1}{2}}(\zeta)$ and $y_l(\zeta) = \sqrt{\frac{\pi}{2\zeta}}\,Y_{l+\frac{1}{2}}(\zeta)$ ...

where $v^2 \equiv \frac{1}{\epsilon\mu}$.

The ratio of C_e to C_h will be determined from the first order Maxwell system. Again one has TM pure conical modes with $\underset{(0)}{\hat{\mathbf{h}}} = 0$ and TE pure conical modes with $\underset{(0)}{\hat{\mathbf{e}}} = 0$. Since the structure of the longitudinal modes has been determined up to a relative scale the remaining fields follow by substitution into (130) and (131). One finds pure (complex) TM modes:

$$\underset{(0)}{\hat{\mathbf{h}}} = 0 \tag{142}$$

$$\underset{(1)}{\hat{\mathbf{h}}} = e^{i\omega t}\,\frac{\omega}{v} r \mathcal{Z}_l\left(\frac{\omega}{v} r\right) \# \boldsymbol{d} Y_l^m \tag{143}$$

$$\underset{(0)}{\hat{\mathbf{e}}} = i\frac{\omega}{\epsilon\, v^2}\, l(l+1) e^{i\omega t} \frac{\mathcal{Z}_l\left(\frac{\omega}{v} r\right)}{\frac{\omega}{v} r}\, Y_l^m \tag{144}$$

$$\underset{(1)}{\hat{\mathbf{e}}} = -\frac{i}{\epsilon v}\, e^{i\omega t}\, \frac{d}{d\zeta}\left(\zeta \mathcal{Z}_l(\zeta)\right)|_{\zeta = \frac{\omega r}{v}}\, \boldsymbol{d} Y_l^m \tag{145}$$

The TM mode spectrum is determined by values of l that ensure the pure conical modes satisfy the appropriate boundary conditions. This can be achieved by requiring $Y_l^m|_{\partial\mathcal{D}} = 0$ or (since one demands periodicity in ϕ)

$$P_l^m(\cos\theta_0) = 0 \tag{146}$$

This transcendental equation has infinitely many (real) roots for each integer m and real value θ_0:

$$l = \hat{l}(m, \theta_0) \tag{147}$$

Similarly one finds pure (complex) TE modes:

$$\underset{(0)}{\hat{\mathbf{e}}} = 0 \tag{148}$$

$$\underset{(1)}{\hat{\mathbf{e}}} = e^{i\omega t}\,\frac{\omega}{v} r\, \mathcal{Z}_L\left(\frac{\omega}{v} r\right)\, \# \boldsymbol{d} Y_L^m \tag{149}$$

$$\underset{(1)}{\hat{\mathbf{h}}} = -i\frac{1}{\mu v}\, e^{i\omega t}\, \frac{d}{d\zeta}\left(\zeta \mathcal{Z}_L(\zeta)\right)|_{\zeta=\frac{\omega r}{v}}\ \boldsymbol{d} Y_L^m \tag{150}$$

$$\underset{(0)}{\hat{\mathbf{h}}} = -i\,\frac{\omega}{\mu v^2} L(L+1)\, e^{i\omega t}\, \frac{\mathcal{Z}_L(\frac{\omega}{v} r)}{\frac{\omega}{v} r}\ Y_L^m \tag{151}$$

For pure TE modes the boundary conditions require $\boldsymbol{d}\theta \wedge \hat{\#} \boldsymbol{d} Y_L^m|_{\partial\mathcal{D}} = 0$ or

$$\frac{d}{d\mu} P_L^m(\mu)|_{\mu=\cos\theta_0} = 0 \tag{152}$$

This transcendental equation also has infinitely many (real) roots for each integer m and real value θ_0:

$$L = \hat{L}(m, \theta_0) \tag{153}$$

Thus one may label the pure conical modes by $N = (m, l)$ and $M = (m, L)$ where l and L are determined by the electromagnetic boundary conditions above. Unlike the cylindrical case there is no simple sharp cut off in ω for propagating modes in the truncated cone with fixed apex angle θ_0. Here the propagation characteristics are determined for each mode by the behaviour of the factor $e^{i\omega t}\mathcal{Z}_l\left(\frac{\omega}{v} r\right)$ or $e^{i\omega t}\mathcal{Z}_L(\frac{\omega}{v} r)$. If a pure conical mode is excited near the narrow end of a truncated cone then the appropriate description will have outgoing ($r \to \infty$) propagating configurations with $\mathcal{Z}_l = h_l^{(2)}$ or $\mathcal{Z}_L = h_L^{(2)}$. A detailed analysis of such a mode reveals that the electromagnetic field pattern does not have radial oscillations for all r. Again this difference from the cylindrical case can be attributed to the lack of translation invariance in the cone. The precise manner in which the radial oscillations begin as a function of r however is controlled by the root l or L and hence by m and the apex angle $2\theta_0$.

A general mixed configuration that may be excited by sources in the cone can now be written in the form (111), (112), (113), (114) but with mode functions (134) and (135) and mode summations over $N = (m, l)$ and $M = (m, L)$. The $V_N^E, V_M^H, I_N^E, I_M^H, \gamma_N^E, \gamma_M^H$ follow by substitution into (119), (120), (122), (123), (126), (127) after projection, using the (complexified) orthogonality properties of the Dirichlet and Neumann functions for $\mathcal{D}$:

$$\int_{\mathcal{D}} \bar{\Phi}_N\, \Phi_M\, \hat{\#}1 = \mathcal{N}_M{}^2 \delta_{MN} \tag{154}$$

$$\int_{\mathcal{D}} \bar{\Psi}_N \, \Psi_M \, \hat{\#} 1 = \mathcal{M}_M{}^2 \delta_{MN} \tag{155}$$

13 Approximation schemes for fields in irregular cylindrical guides

13.1 Scalar fields in a tapered cylinder without sources

In the above, attention has been concentrated on regular guides. The particular geometrical properties of such structures ensures that they are amenable to a global analysis in terms of functions with established properties. For guides with irregular geometric properties this is rarely possible. If the irregularity is in some sense a localised variation in an otherwise regular structure approximation methods may be applicable. Since the vector nature of the electromagnetic field adds an additional layer of complexity to such methods it is useful to consider first localised modifications to a scalar field subject to Dirichlet boundary conditions at the surface of an irregular guide. Consider then an irregularly tapered cylinder which in cylindrical coordinates (ρ, ϕ, z) has as its surface of revolution about the z axis the form $\rho = a\,\chi(z)$ for some function χ and constant a. For the immediate analysis it is assumed that χ is smoothly varying in the vicinity of $z = 0$. It will be called a *surface perturbation* if additionally it takes the form

$$\chi(z) = 1 + \epsilon \hat{\chi}(z) \tag{156}$$

where in this section $\epsilon << 1$ denotes a small parameter.

Suppose the amplitude of the harmonic scalar field $e^{i\omega\, t}\Phi(\rho, \phi, z)$ is required to satisfy

$$\boldsymbol{d} \# \boldsymbol{d} \, \Phi + (\frac{\omega}{c})^2 \, \Phi \# 1 = 0 \tag{157}$$

for $0 \leq \phi \leq 2\pi, \; -\infty \leq z \leq \infty, \; 0 \leq \rho \leq a\chi(z)$ and

$$\Phi(a\,\chi(z), \phi, z) = 0 \tag{158}$$

For a general $\chi(\zeta)$ it is unlikely that (157) will have exact solutions that are products of functions of a single coordinate or are expressible in terms of known functions.

One can use the freedom to change coordinates to "straighten out the taper". Thus (with c=1) pass from the chart with coordinates ρ, ϕ, z to r, θ, ζ with the transformation:

$$\rho = r\,\chi(\zeta), \qquad \phi = \theta, \qquad z = \zeta \tag{159}$$

Then the local coframe $\{e^1 = \boldsymbol{d}\,\rho,\ e^2 = \rho\boldsymbol{d}\,\phi,\ e^3 = \boldsymbol{d}\,z\}$ in the "physical chart" becomes $\{f^1 = \boldsymbol{d}\,(r\chi(\zeta)), f^2 = r\,\chi(\zeta)\boldsymbol{d}\,\theta,\ f^3 = \boldsymbol{d}\,\zeta$ in the new chart. Furthermore the surface $\rho = a\chi(z)$ is $r = a$ in the new chart and one is confronted with solving:

$$\boldsymbol{d}\,\#\boldsymbol{d}\,Q + \omega^2\,Q\#1 = 0 \tag{160}$$

for $Q(r,\theta,\zeta) = \Phi(\rho,\phi,z)$ in the region $0 \leq \theta \leq 2\pi,\ -\infty \leq \zeta \leq \infty,\ 0 \leq r \leq a$ subject to

$$Q(a,\theta,\zeta) = 0 \tag{161}$$

The structure equations for the coframe in the new coordinates are now:

$$\boldsymbol{d}\,f^1 = 0, \qquad \boldsymbol{d}\,f^2 = \frac{1}{r\chi(\zeta)} f^1 \wedge f^2, \qquad \boldsymbol{d}\,f^3 = 0 \tag{162}$$

and the tensor $\underline{g}$ takes the form

$$\underline{g} = \sum_{i=1}^{3} f^i \otimes f^i$$

Hence the $3-D$ Laplacian $-\boldsymbol{d}\,\#\boldsymbol{d}$ has a representation in the (r,θ,ζ) chart that depends on χ and its derivatives. However it is only the ζ dependence that makes it different from the regular case where $\epsilon = 0$. So since Q must be periodic in θ and regular at $r = 0$ one may write (with m integer):

$$Q(r,\theta,\zeta) = \sum_{m=-\infty}^{\infty} \sum_{p} \mathcal{C}_{mp}(\zeta) J_m\left(x_{mp}\frac{r}{a}\right) e^{im\theta} \tag{163}$$

for some functions $\mathcal{C}_{mp}(\zeta)$ to be determined. Clearly when $\epsilon = 0$ one has the special (complex) solutions ${\mathcal{C}_{mp}}^0(\zeta) = \exp(i\, k_{mp}\zeta)$ where $k^2_{mp} = \omega^2 - (\frac{x_{mp}}{a})^2$. Note that, with $J_m(x_{mp}) = 0$, Q is constructed to satisfy the required boundary condition at $r = a$. For $\epsilon \neq 0$, Q must satisfy (160):

$$\frac{1}{r^2\chi^2}\,\partial^2_{\theta\theta}Q - 2\epsilon r\frac{\hat{\chi}'}{\chi}\,\partial^2_{r\zeta}\,Q + \partial^2_{\zeta\zeta}Q + \Lambda(r,\zeta)\,\partial^2_{rr}Q + H(r,\zeta)\,\partial_r Q + \omega^2\,Q = 0 \tag{164}$$

where

$$\Lambda(r,\zeta) = \frac{1 + r^2\epsilon^2\hat{\chi}'^2}{\chi^2} \tag{165}$$

$$H(r,\zeta) = 2\epsilon^2 r\left(\frac{\hat{\chi}}{\chi}\right)^2 - \epsilon r\frac{\hat{\chi}''}{\chi} + \frac{1}{r\chi^2} \tag{166}$$

A first order perturbative approach *in the new* (r, θ, ζ) chart is suggested with

$$\mathcal{C}_{mp}(\zeta) = {\mathcal{C}_{mp}}^0(\zeta) + \epsilon\,{\mathcal{C}_{mp}}^1(\zeta) \tag{167}$$

Suppose one takes for the zeroth approximation ($\epsilon = 0$) the real solution with ${\mathcal{C}_{mp}}^0(\zeta) = A_{mp}\sin(k_{mp}\zeta)$ for some constant A_{mp}.

Since χ is independent of θ one seeks a system of ordinary differential equations for ${\mathcal{C}_{mp}}^1(\zeta)$ for each allowed integer m but coupled by the mode index p.

A simple example will illustrate the additional technology required to extricate this system. Suppose one has $\hat{\chi}(\zeta) = \zeta$. [10]

Inserting (163) into (160) or (164) one finds that for all allowed m, p and all r, ζ the amplitudes ${\mathcal{C}_{mp}}^1(\zeta)$ must satisfy to first order in ϵ:

$$\sum_m e^{im\theta}\sum_p\left(\beta^m_p(\zeta)J_m\left(x_{mp}\frac{r}{a}\right) + r\,\alpha^m_p(\zeta)\,J_{m+1}\left(x_{mp}\frac{r}{a}\right)\right) = 0 \tag{168}$$

[10] This of course will generate a cone and a global analysis of this case can be deduced from the analysis in earlier sections above. Here it is for illustrative purposes alone and should be regarded in the spirit of a perturbation in the vicinity of $z = 0$.

where

$$\alpha_p^m(\zeta) = \frac{2x_{mp}}{a} A_{mp} k_{mp} \cos(k_{mp}\zeta)$$

and

$$\beta_p^m(\zeta) = \Big(\mathcal{C}_{mp}^{1\prime\prime}(\zeta) + (\omega^2 - \tfrac{x_{mp}^2}{a^2})\, \mathcal{C}_{mp}^1 + 2\zeta A_{mp} \tfrac{x_{mp}^2}{a^2} \sin(k_{mp}\zeta) -$$

$$2\, A_{mp} m k_{mp} \cos(k_{mp}\zeta)\big)$$

All derivatives of the Bessel function have be re-expressed in terms of Bessel functions using standard Bessel relations. Since the x_{mp} are roots satisfying $J_m(x_{mp}) = 0$ one can then use the classical relations:

$$\int_0^a J_m\left(x_{mp}\frac{r}{a}\right) J_m(x_{mp'}\frac{r}{a})\, r\, dr = \delta_{pp'} N_p^m \tag{169}$$

where $N_p^m = \frac{1}{2} J_{m+1}^2(x_{mp})$. Thus applying the operation $\int_0^a r\, dr\, J_m(x_{mq}\frac{r}{a})$ to the equation (168) yields:

$$\sum_m e^{im\theta} \sum_p \left(\beta_p^m(\zeta)\delta_{pq} N_q^m + \alpha_p^m(\zeta) M_{pq}^m\right) = 0 \tag{170}$$

where

$$M_{pq}^m \equiv \int_0^a r^2\, d\, r J_m\left(x_{mq}\frac{r}{a}\right)\, J_{m+1}\left(x_{mp}\frac{r}{a}\right) \tag{171}$$

Since the $\{e^{im\theta}\}$ are complete with

$$\frac{1}{2\pi}\int_0^{2\pi} e^{im\theta}\, d\theta = \delta_{0m}$$

this requires

$$\beta_q^m(\zeta) + \frac{1}{N_q^m} \sum_p \alpha_p^m(\zeta)\, M_{pq}^m = 0 \tag{172}$$

where $m = 0, \pm 1, \pm 2, \ldots$ and p, q range over root labels. Thus for each integer m one has to solve at this order in ϵ an infinitely coupled system of second order o.d.e.'s for ${\mathcal{C}_{mp}}^1(\zeta)$ in terms of the amplitudes A_{mp} associated with the zeroth order solutions. A natural requirement to fix the solution is that ${\mathcal{C}_{mp}}^1(\zeta) = 0$ when ${\mathcal{C}_{mp}}^0(\zeta) = 0$. The nature of the (p, q) coupling is governed by the matrix element M^m_{pq} for each m. A further simplification is to truncate the sum over p. In particular if one keeps just the term with $p = q$ then one gets a correction to one of the regular cylindrical modes produced by the linear taper. The resulting o.d.e. can be readily solved in terms of trigonometric functions. In terms of the original physical coordinates (ρ, ϕ, z) such first order corrected modes take the form:

$$\Phi(\rho, \phi, z) \simeq \left\{ A_{mp} \sin(k_{mp} z) + \epsilon\, {\mathcal{C}_{mp}}^1(\zeta) \right\} \, J_m\left(\frac{x_{mp}}{a} \frac{\rho}{\chi(z)} \right) e^{im\phi} \tag{173}$$

corresponding to a modified standing wave. This might be thought of as follows. Suppose the time harmonic scalar standing wave amplitude

$$\Phi_0(\rho, \phi, z) = A_{mp} \sin(k_{mp} z) \, J_m\left(\frac{x_{mp}}{a} \rho \right) \, e^{im\phi} \tag{174}$$

is established in a regular cylinder of radius a. If the tube is slowly tapered to have the surface of revolution $r = a\,\chi(\zeta)$ then for small ϵ the standing wave distorts to the approximate form $\Phi(\rho, \phi, z)$ above. Note that this is a first order in ϵ correction in the (r, θ, ζ) chart (where the boundary conditions are easy to apply) but not in the physical (ρ, ϕ, z) chart.

13.2 Maxwell fields in a tapered cylinder with internal sources

A direct application of the method above to the Maxwell system of coupled vector fields leads to further coupling between different components of the vector fields. However a process of "diagonalisation" is afforded by exploiting the orthogonality of the appropriate Dirichlet and Neumann modes. This will now be made explicit for the irregular cylinder and generalises the previous discussion of fields excited by sources in the perfectly conducting cylinder. The methods are however readily applicable to other geometries.

In the (r, θ, ζ) chart we have introduced the coframe.

$$\{f^1 = \boldsymbol{d}\,(r\chi(\zeta)), \quad f^2 = r\,\chi(\zeta)\boldsymbol{d}\,\theta, \quad f^3 = \boldsymbol{d}\,\zeta\} \tag{175}$$

It is also convenient to introduce in this chart the coframe

$$\{\breve{e}^1 = \boldsymbol{d}\,r, \quad \breve{e}^2 = r\,\boldsymbol{d}\,\theta, \quad \breve{e}^3 = \boldsymbol{d}\,\zeta\} \tag{176}$$

These two frames coincide when $\epsilon = 0$ so, for small ϵ, (175) is a perturbation of frame (176) and

$$f^1 = \chi(\zeta)\breve{e}^1 + r\chi'(\zeta)\,\breve{e}^3, \quad f^2 = \chi(\zeta)\,\breve{e}^2, \quad f^3 = \breve{e}^3 \tag{177}$$

Now we have the Hodge map $\#$ associated with the orthonormal coframe $\{f^1, f^2, f^3\}$ so introduce the "unperturbed"Hodge map $\breve{\#}$ associated with the orthonormal unperturbed coframe $\{\breve{e}^1, \breve{e}^2, \breve{e}^3\}$. Thus

$$\begin{aligned}
\#1 &= f^1 \wedge f^2 \wedge f^3 = \hat{\#}1 \wedge \boldsymbol{d}\,\zeta \\
\hat{\#}1 &= f^1 \wedge f^2 \\
\breve{\#}1 &= \breve{e}^1 \wedge \breve{e}^2 \wedge \breve{e}^3 = \breve{\hat{\#}}1 \wedge \boldsymbol{d}\,\zeta \\
\breve{\hat{\#}}1 &= \breve{e}^1 \wedge \breve{e}^2 = r\,\boldsymbol{d}\,r \wedge \boldsymbol{d}\,\theta
\end{aligned}$$

Guided by the need to apply boundary conditions in this chart introduce the Dirichlet and Neumann mode functions satisfying

$$\boldsymbol{d}\,\breve{\hat{\#}}\boldsymbol{d}\,\Phi_N(r,\theta) = -\beta_N^2\,\Phi_N(r,\theta)\,r\,\boldsymbol{d}\,r \wedge \boldsymbol{d}\,\theta \tag{178}$$

$$\boldsymbol{d}\,\breve{\hat{\#}}\boldsymbol{d}\,\Psi_M(r,\theta) = -\alpha_M^2\,\Psi_M(r,\theta)r\,\boldsymbol{d}\,r \wedge \boldsymbol{d}\,\theta \tag{179}$$

in the domain $\mathcal{D}$ defined by $0 \le r \le a,\ 0 \le \theta \le 2\pi$. It is straightforward to verify that

$$\#\boldsymbol{d}\,r = \big(1 + r^2\,(\chi')^2\big)\breve{\#}\boldsymbol{d}\,r - r\,\chi\chi'\breve{\#}\boldsymbol{d}\,\zeta \tag{180}$$

$$\#\boldsymbol{d}\,\theta = \breve{\#}\boldsymbol{d}\,\theta \tag{181}$$

and $\breve{\#}\boldsymbol{d}\,r = r\boldsymbol{d}\,\theta \wedge \boldsymbol{d}\,\zeta$, $\breve{\#}\boldsymbol{d}\,\zeta = r\,\boldsymbol{d}\,r \wedge \boldsymbol{d}\,\theta$. Hence for any scalar $\psi(r,\theta)$

$$\#\boldsymbol{d}\,\psi = \breve{\#}\boldsymbol{d}\,\psi + (\frac{\partial}{\partial r}\psi)\,\big((r\,\chi')^2 r\,\boldsymbol{d}\,\theta \wedge \boldsymbol{d}\,\zeta - \chi\chi'\,r^2\,\boldsymbol{d}\,r \wedge \boldsymbol{d}\,\theta\big) \tag{182}$$

where

$$\breve{\#}\boldsymbol{d}\,\psi = \breve{\#}\left(\frac{\partial}{\partial r}\psi\,\boldsymbol{d}\,r + \frac{\partial}{\partial \theta}\psi\,\boldsymbol{d}\,\theta\right) = (\breve{\ddot{\#}}\boldsymbol{d}\,\psi) \wedge \boldsymbol{d}\,\zeta \tag{183}$$

and

$$\#(\boldsymbol{d}\,\zeta \wedge \boldsymbol{d}\,\psi) = \breve{\ddot{\#}}\boldsymbol{d}\,\psi - \frac{\chi'}{\chi}\frac{\partial}{\partial\theta}\psi\;\boldsymbol{d}\,\zeta \tag{184}$$

In terms of $\boldsymbol{d}\,\zeta$

$$\#\boldsymbol{d}\,\psi = (\breve{\ddot{\#}}\boldsymbol{d}\,\psi) \wedge \boldsymbol{d}\,\zeta + (\frac{\partial}{\partial r}\psi)\,\big\{(r\,\chi')^2\big\}\,r\,\boldsymbol{d}\,\theta \wedge \boldsymbol{d}\,\zeta - (\frac{\partial}{\partial r}\psi)\,r\chi\chi'\breve{\#}1 \tag{185}$$

and so with (178), (179)

$$\begin{aligned}\boldsymbol{d}\,\#(\boldsymbol{d}\,\zeta \wedge \boldsymbol{d}\,\Psi_M) = -\alpha_M^2\,\Psi_M\,r\,\boldsymbol{d}\,r \wedge \boldsymbol{d}\,\theta - \frac{\chi'}{\chi}\frac{\partial}{\partial r}\frac{\partial}{\partial\theta}\Psi_M\,\boldsymbol{d}\,r \wedge \boldsymbol{d}\,\zeta \\ -\frac{\chi'}{\chi}\frac{\partial}{\partial\theta}\frac{\partial}{\partial\theta}\Psi_M\,\boldsymbol{d}\,\theta \wedge \boldsymbol{d}\,\zeta\end{aligned} \tag{186}$$

$$\begin{aligned}\boldsymbol{d}\,\#(\boldsymbol{d}\,\zeta \wedge \boldsymbol{d}\,\Phi_N) = -\beta_N^2\,\Phi_N\,r\,\boldsymbol{d}\,r \wedge \boldsymbol{d}\,\theta - \frac{\chi'}{\chi}\frac{\partial}{\partial r}\frac{\partial}{\partial\theta}\Phi_N\,\boldsymbol{d}\,r \wedge \boldsymbol{d}\,\zeta \\ -\frac{\chi'}{\chi}\frac{\partial}{\partial\theta}\frac{\partial}{\partial\theta}\Phi_N\,\boldsymbol{d}\,\theta \wedge \boldsymbol{d}\,\zeta\end{aligned} \tag{187}$$

$$\begin{aligned}\boldsymbol{d}\,\#\boldsymbol{d}\,\Psi_M \;=\; & -\big(\alpha_M^2\,r\Psi_M\,-\partial_r(r^3\partial_r\Psi_M)(\chi')^2\;+ \\ & r^2\,(\partial_r\Psi_M)\partial_\zeta(\chi\chi')\big)\,\boldsymbol{d}\,\zeta \wedge \boldsymbol{d}\,r \wedge \boldsymbol{d}\,\theta\end{aligned} \tag{188}$$

$$\boldsymbol{d}\#\boldsymbol{d}\,\Phi_N = -\big(\beta_N^2\, r\Phi_N - \partial_r(r^3\partial_r\Phi_N)(\chi')^2 + r^2\,(\partial_r\Phi_N)\partial_\zeta(\chi\chi')\big)\,\boldsymbol{d}\,\zeta\wedge\boldsymbol{d}\,r\wedge\boldsymbol{d}\,\theta \tag{189}$$

These formulae express the manner in which expressions involving $\#$ (particularly the Laplacians) are perturbed when $\chi\neq 1$ and will be used in the development below.

The current source is the 2−form $\underset{(2)}{J}$ where $\underset{(2)}{J} = \underset{(1)}{\hat{J}}\wedge\boldsymbol{d}\,\zeta + \underset{(0)}{\hat{J}}\,\#1$. In the $\{t,z,\rho,\phi\}$ spacetime chart write:

$$\underset{(1)}{\hat{J}} = J_\rho(t,z,\rho,\phi)\,\boldsymbol{d}\,\rho + J_\phi(t,z,\rho,\phi)\,\boldsymbol{d}\,\phi \tag{190}$$

and in the $\{t,\zeta,r,\theta\}$ chart (see (159)) let:

$$S_\rho(t,\zeta,r,\theta) = J_\rho(t,z,\rho,\phi) \tag{191}$$

$$S_\theta(t,\zeta,r,\theta) = J_\phi(t,z,\rho,\phi) \tag{192}$$

$$S_0(t,\zeta,r,\theta) = J_0(t,z,\rho,\phi) \tag{193}$$

so the current 2-form has the representation

$$\underset{(2)}{S}(t,\zeta,r,\theta) = \Big(S_\rho(t,\zeta,r,\theta)\,\boldsymbol{d}\,\big(r\chi(\zeta)\big) + S_\theta(t,\zeta,r,\theta)\,\boldsymbol{d}\,\theta\Big)\wedge\boldsymbol{d}\,\zeta + S_0(t,\zeta,r,\theta)\,r\,\chi(\zeta)\,\boldsymbol{d}\,\big(r\chi(\zeta)\big)\wedge\boldsymbol{d}\,\theta \tag{194}$$

Let the scalar source charge density in the (t,z,r,θ) chart be denoted $\tilde{\rho}(t,z,r,\theta)$.

The methodology now is to take the Maxwell field expansions

$$\underset{(1)}{\mathbf{e}} = \sum_N V_N^E(\epsilon,t,\zeta)\,\boldsymbol{d}\,\Phi_N + \sum_M V_M^H(\epsilon,t,\zeta)\#(\boldsymbol{d}\,\zeta\wedge\boldsymbol{d}\,\Psi_M) + \sum_N \gamma_N^E(\epsilon,t,\zeta)\,\Phi_N\boldsymbol{d}\,\zeta \tag{195}$$

$$\underset{(1)}{\mathbf{h}} = \sum_N I_N^E(\epsilon, t, \zeta)\,\#(\boldsymbol{d}\,\zeta \wedge \boldsymbol{d}\,\Phi_N) + \sum_M I_M^H(\epsilon, t, \zeta)\boldsymbol{d}\,\Psi_M + \sum_M \gamma_M^H(\epsilon, t, \zeta)\Psi_M\,\boldsymbol{d}\,\zeta \tag{196}$$

appropriate for the boundary conditions that are accommodated by the mode functions $\Phi_N(r, \theta)$ and $\Psi_M(r, \theta)$ and substitute into the Maxwell system[11]

$$\boldsymbol{d}\,\underset{(1)}{\mathbf{e}} + \mu\#\underset{(1)}{\dot{\mathbf{h}}} = 0 \tag{197}$$

$$\boldsymbol{d}\,\underset{(1)}{\mathbf{h}} - \mu\mathcal{Y}^2\#\underset{(1)}{\dot{\mathbf{e}}} - \underset{(2)}{J} = 0 \tag{198}$$

$$\boldsymbol{d}\#\underset{(1)}{\mathbf{h}} = 0 \tag{199}$$

$$\mu\mathcal{Y}^2\boldsymbol{d}\#\underset{(1)}{\mathbf{e}} - \tilde{\rho}\#1 = 0, \tag{200}$$

simplify the results using the relations above and project into $0-$form longitudinal equations (coefficients of $\boldsymbol{d}\,\zeta$), $0-$ form equations (coefficients of $\#1$) and transverse $1-$form equations (independent of $\boldsymbol{d}\,\zeta$). All the ϵ dependence resides in the amplitudes $V_N^E, I_M^H, \gamma_N^E, \gamma_M^H$ and the shape perturbation χ and their derivatives. The (r, θ) dependence of the resulting system is therefore made explicit and by judicious use of the mode orthogonality relations an over-determined coupled p.d.e. system for the amplitudes can be calculated in terms of the prescribed sources. If $\epsilon << 1$ one can effect a perturbative approach to this system.

Following this procedure one finds after some calculation that to zeroth order:

$$V_N^{E(0)\prime} + \mu\dot{I}_N^{E(0)} - \gamma_N^{E(0)} = 0 \tag{201}$$

$$V_N^{H(0)\prime} - \mu\dot{I}_N^{H(0)} = 0 \tag{202}$$

[11] Since ϵ is here a perturbation parameter the permittivity of the medium will henceforth be denoted by $\mu\mathcal{Y}^2$ in terms of the medium characteristic admittance $\mathcal{Y}$. Similarly the 0-form $\tilde{\rho}$ is used for the charge density in order to distinguish it from the coordinate ρ.

$$V_N^{H(0)}\alpha_N^2 - \mu\dot{\gamma}_N^{H(0)} = 0 \tag{203}$$

$$(I_N^{E(0)\prime} + \mu\mathcal{Y}^2\dot{V}_N^{E(0)})\mathcal{N}_N^2\beta_N^2 - \int_{\mathcal{D}} S_\rho\, \boldsymbol{d}\, r \wedge \boldsymbol{d}\, \Phi_N - \int_{\mathcal{D}} S_\theta\, \boldsymbol{d}\, \theta \wedge \boldsymbol{d}\, \Phi_N = 0 \tag{204}$$

$$\begin{aligned}(I_N^{H(0)\prime} - \mu\mathcal{Y}^2\dot{V}_N^{H(0)} - \gamma_N^{H(0)})\mathcal{M}_N^2\alpha_N^2 - \textstyle\int_{\mathcal{D}} S_\rho\, r\, \boldsymbol{d}\, \theta \wedge \boldsymbol{d}\, \Psi_N + \\ \textstyle\int_{\mathcal{D}} S_\theta\, \boldsymbol{d}\, r \wedge \boldsymbol{d}\, \Psi_N = 0\end{aligned} \tag{205}$$

$$(\mu\mathcal{Y}^2\dot{\gamma}_N^{E(0)} + I_N^{E(0)}\beta_N^2)\mathcal{N}_N^2 + \int_{\mathcal{D}} S_0\, \Phi_N\, r\, \boldsymbol{d}\, r \wedge \boldsymbol{d}\, \theta = 0 \tag{206}$$

$$I_N^{H(0)}\alpha_N^2 - \gamma_N^{H(0)\prime} = 0 \tag{207}$$

$$(\gamma_N^{E(0)\prime} - V_N^{E(0)}\beta_N^2)\mu\mathcal{Y}^2\mathcal{N}_N^2 - \int_{\mathcal{D}} \tilde{\rho}\, \Phi_N\, r\, \boldsymbol{d}\, r \wedge \boldsymbol{d}\, \theta = 0 \tag{208}$$

and to first order

$$\begin{aligned}&\big(V_N^{E(1)\prime} - \gamma_N^{E(1)} + \mu\dot{I}_N^{E(1)}\big)\beta_N^2\mathcal{N}_N^2 + \\ &\textstyle\sum_M \big(- V_M^{H(0)}\hat{\chi} \int_{\mathcal{D}} (\partial^2_{r\theta}\Psi_M)\, r\boldsymbol{d}\; r \wedge \boldsymbol{d}\, \Phi_N \\ &+ V_M^{H(0)}\hat{\chi} \int_{\mathcal{D}} \frac{1}{r}(\partial^2_{\theta\theta}\Psi_M \boldsymbol{d}\, \theta \wedge \boldsymbol{d}\, \Phi_N + \mu\, \dot{\gamma}_M^{H(0)}\hat{\chi}' \int_{\mathcal{D}} \Psi_M\, r\boldsymbol{d}\, \theta \wedge\; \boldsymbol{d}\, \Phi_N\big) = 0\end{aligned} \tag{209}$$

$$\begin{aligned}&\big(V_N^{H(1)\prime} - \mu\, \dot{I}_N^{H(1)}\big)\alpha_N^2\mathcal{M}_N^2 - \textstyle\sum_M \big(V_M^{H(0)}\hat{\chi} \int_{\mathcal{D}} \partial^2_{r\theta}\Psi_M\, \boldsymbol{d}\, r \wedge \boldsymbol{d}\, \Psi_N \\ &+ V_M^{H(0)}\hat{\chi} \textstyle\int_{\mathcal{D}} \partial^2_{\theta\theta}\Psi_M\, \boldsymbol{d}\, \theta \wedge \boldsymbol{d}\, \Psi_N + \mu\, \dot{\gamma}_M^{H(0)}\hat{\chi}' \int_{\mathcal{D}} r^2\Psi_M\, \boldsymbol{d}\, \theta \wedge \boldsymbol{d}\, \Psi_N\big) = 0\end{aligned} \tag{210}$$

$$\begin{aligned}&(\mu\dot{\gamma}_N^{H(1)} + 2\mu\dot{\gamma}_N^{H(0)}\hat{\chi} - V_N^{H(1)}\alpha_N^2)\mathcal{M}_N^2 - \\ &\qquad \mu\, \hat{\chi}'\dot{I}_N^{H(0)} \textstyle\sum_M \int_{\mathcal{D}} \Psi_N(\partial_r\Psi_M)r^2\, \boldsymbol{d}\, r \wedge \boldsymbol{d}\, \theta = 0\end{aligned} \tag{211}$$

$$
\begin{aligned}
(I_N^{E(1)\prime} + \mu\mathcal{Y}^2\dot{V}_N^{E(1)})\beta_N^2\mathcal{N}_N^2 + \sum_M \big(- I_M^{E(0)}\hat{\chi}' \int_{\mathcal{D}} (\partial^2_{r\theta}\Phi_M)\boldsymbol{d}\,r\wedge\boldsymbol{d}\,\Phi_N \\
-\hat{\chi}\int_{\mathcal{D}} S_\rho\,\boldsymbol{d}\,r\wedge\boldsymbol{d}\,\Phi_N - I_M^{E(0)}\hat{\chi}'\int_{\mathcal{D}}(\partial^2_{\theta\theta}\Phi_M)\boldsymbol{d}\,\theta\wedge\boldsymbol{d}\,\Phi_N \\
+\mu\mathcal{Y}^2\dot{\gamma}_M^{E(0)}\hat{\chi}'\int_{\mathcal{D}}\Phi_M\,r^2\boldsymbol{d}\,\theta\wedge\boldsymbol{d}\,\Phi_N + \hat{\chi}'\int_{\mathcal{D}} S_0\,r^2\boldsymbol{d}\,\theta\wedge\boldsymbol{d}\,\Phi_N\big) = 0
\end{aligned}
\tag{212}
$$

$$
\begin{aligned}
&(I_N^{H(1)\prime} - \mu\mathcal{Y}^2\dot{V}_N^{H(1)} - \gamma_N^{H(1)})\alpha_N^2\mathcal{M}_N^2 + \\
&\textstyle\sum_M \big(- I_M^{E(0)}\hat{\chi}\int_{\mathcal{D}}\partial^2_{r\theta}\Phi_M\,r\,\boldsymbol{d}\,\theta\wedge\boldsymbol{d}\,\Psi_N \\
&I_M^{E(0)}\hat{\chi}\textstyle\int_{\mathcal{D}}\frac{1}{r}\partial^2_{\theta\theta}\Phi_M\,\boldsymbol{d}\,r\wedge\boldsymbol{d}\,\Psi_N - \mu\mathcal{Y}^2\dot{\gamma}_M^{(0)}\hat{\chi}'\int_{\mathcal{D}}\Phi_M\,r\,\boldsymbol{d}\,r\wedge\boldsymbol{d}\,\Psi_N - \\
&\hat{\chi}'\textstyle\int_{\mathcal{D}} S_0\,r\,\boldsymbol{d}\,r\wedge\boldsymbol{d}\,\Psi_N - \hat{\chi}\int_{\mathcal{D}} S_\rho\,r\,\boldsymbol{d}\,\theta\wedge\boldsymbol{d}\,\Psi_N\big) = 0
\end{aligned}
\tag{213}
$$

$$
\begin{aligned}
(2\mu\mathcal{Y}^2\dot{\gamma}_N^{E(0)}\hat{\chi} + \mu\mathcal{Y}^2\dot{\gamma}_N^{E(1)} + I_N^{E(1)}\beta_N^2)\mathcal{N}_N^2 + 2\hat{\chi}\int_{\mathcal{D}} S_0\Phi_N\,r\,\boldsymbol{d}\,r\wedge\boldsymbol{d}\,\theta \\
-\mu\mathcal{Y}^2\sum_M \dot{V}_M^{E(0)}\hat{\chi}'\int_{\mathcal{D}}\Phi_N\,(\partial_r\Phi_M)\,r^2\,\boldsymbol{d}\,r\wedge\boldsymbol{d}\,\theta = 0
\end{aligned}
\tag{214}
$$

$$
\begin{aligned}
(I_N^{H(1)}\alpha_N^2 - 2\gamma_N^{H(0)\prime}\hat{\chi} - \gamma_N^{H(1)\prime})\mathcal{M}_N^2 \\
+\sum_M \big(\hat{\chi}' I_M^{H(0)\prime} + \hat{\chi}'\gamma_M^{H(0)} + \hat{\chi}'' I_M^{H(0)}\big)\int_{\mathcal{D}}\Psi_N(\partial_r\Psi_M)\,r^2\,\boldsymbol{d}\,r\wedge\boldsymbol{d}\,\theta = 0
\end{aligned}
\tag{215}
$$

$$
\begin{aligned}
(-V_N^{E(1)}\beta_N^2 + 2\gamma_N^{E(0)\prime}\hat{\chi} + \gamma_N^{E(1)})\mathcal{N}_N^2 \\
-\sum_M \big\{(V_M^{E(0)\prime}\hat{\chi}' + V_M^{E(0)}\hat{\chi}'' + \gamma_M^{E(0)}\hat{\chi}')\int_{\mathcal{D}}\Phi_N(\partial_r\Phi_M)\,r\,\boldsymbol{d}\,r\wedge\boldsymbol{d}\,\theta\big\} \\
-2\frac{\hat{\chi}}{\mu\mathcal{Y}^2}\int_{\mathcal{D}}\tilde{\rho}\,\Phi_N\,r\,\boldsymbol{d}\,r\wedge\boldsymbol{d}\,\theta = 0
\end{aligned}
\tag{216}
$$

where $V_N^E = V_N^{E(0)} + \epsilon V_N^{E(1)} + \ldots$ etc.

Given the sources one must solve the zeroth order system for the fields $V_N^{E(0)}$, $V_N^{H(0)}$, $I_N^{E(0)}$, $I_N^{H(0)}$, $\gamma_N^{E(0)}$, $\gamma_N^{H(0)}$. These together with the sources and the shape function $\hat{\chi}$ are then used in the first order system to determine the corrections $V_N^{E(1)}$, $V_N^{H(1)}$, $I_N^{E(1)}$, $I_N^{H(1)}$, $\gamma_N^{E(1)}$, $\gamma_N^{H(1)}$. These systems imply constraints since the sources must be compatible with 4−current conservation $d\,j = 0$. For a regular cylinder one has $\hat{\chi}(\zeta) = 0$ and for a source-free (irregular or regular) cylinder $\tilde{\rho} = 0, S_\rho = 0, S_\theta = 0, S_0 = 0$.

By adapting the coframe to other geometries and using the appropriate Dirichlet and Neumann mode functions the above methodology can be readily adapted to other irregular confining domains such as those based on the conical and sector guides discussed earlier.

14 Maxwell fields in a toroid without internal sources

The effects of slight bending and twisting of a guide in space can also be estimated by these perturbative techniques. Suppose a guide with a circular cross section (with radius a) is not straight. Let each cross-section centre trace out a space curve with Frenet curvature $\kappa(z)$ and torsion $\mathcal{T}(z)$ where z denotes arc-length along this space-curve. If the curvature is not too large one can then use a Frenet frame based on this curve to set up a coordinate system $\{r, \theta, z\}$ for the interior of the guide. If the space-curve is given parametrically in terms of the Euclidean position vector $\mathbf{C}(z)$ then points in the interior have Euclidean positions:

$$\mathbf{r} = \mathbf{C}(z) + r\cos\theta\,\mathbf{n} + r\sin\theta\,\mathbf{b} \tag{217}$$

where $\mathbf{n}$ is the Frenet normal and $\mathbf{b}$ is the Frenet bi-normal to the space-curve. A convenient orthonormal coframe for the interior domain is then

$$\begin{aligned} \{e^1 &= \boldsymbol{d}\,x_1 - x_2\mathcal{T}(z)\boldsymbol{d}\,z, \quad e^2 = \boldsymbol{d}\,x_2 + x_1\mathcal{T}(z)\boldsymbol{d}\,z, \\ e^3 &= \big(1 - \kappa(z)\,x_1\big)\,\boldsymbol{d}\,z\} \end{aligned} \tag{218}$$

where $x_1 = r\cos\theta$ and $x_2 = r\sin\theta$. Further details on Frenet coordinates can be found in [5].

To illustrate the effects of curvature consider the perturbation on the sourceless modes produced by deforming a long cylinder into a toroid with small constant curvature $\kappa_0 \equiv \frac{2}{R}$. Thus the fiducial space-curve in this case has zero torsion $\mathcal{T}$. One may use the same Dirichlet and Neumann modes as for the cylinder since the metallic surface of the toroid is $r = a$. Inserting the expansions above for $\underset{(1)}{\mathbf{e}}$ and $\underset{(1)}{\mathbf{h}}$ in the Maxwell system in the absence of sources and noting

$$\#(\boldsymbol{d}\, z \wedge \mathbf{d}\Psi_N) = \hat{\#}\mathbf{d}\Psi_N/(1 - \epsilon\kappa_0 r\cos\theta)$$

$$\#(\boldsymbol{d}\, z \wedge \mathbf{d}\Phi_N) = \hat{\#}\mathbf{d}\Phi_N/(1 - \epsilon\kappa_0 r\cos\theta)$$

$$\#\boldsymbol{d}\,\Psi_N = (1 - \epsilon\kappa_0 r\cos\theta)\ (\hat{\#}\mathbf{d}\Psi_N) \wedge \boldsymbol{d}\ z$$

$$\#\boldsymbol{d}\,\Phi_N = (1 - \epsilon\kappa_0 r\cos\theta)\ (\hat{\#}\mathbf{d}\Phi_N) \wedge \boldsymbol{d}\ z$$

$$\boldsymbol{d}\,\#\boldsymbol{d}\,\Psi_N = -(1-\epsilon\kappa_0 r\cos\theta)\alpha_N^2\Psi_N\, r\, \boldsymbol{d}\, r\wedge\boldsymbol{d}\,\theta\wedge\boldsymbol{d}\, z - (\hat{\#}\mathbf{d}\Psi_N)\wedge\mathbf{d}\left((1-\epsilon\kappa_0 r\cos\theta)\,\boldsymbol{d}\, z\right)$$

$$\boldsymbol{d}\,\# d\Phi_N = -(1-\epsilon\kappa_0 r\cos\theta)\beta_N^2\Phi_N\, r\, \boldsymbol{d}\, r\wedge\boldsymbol{d}\,\theta\wedge\boldsymbol{d}\, z - (\hat{\#}\mathbf{d}\Phi_N)\wedge\mathbf{d}\left((1-\epsilon\kappa_0 r\cos\theta)\,\boldsymbol{d}\, z\right)$$

one finds, to zero and first order in ϵ:

$$\sum_N \left(V_N^{E(0)\prime} + \mu \dot{I}_N^{E(0)} - \gamma_N^{E(0)}\right)\boldsymbol{d}\,\Phi_N + \left(V_N^{H(0)\prime} - \mu\dot{I}_N^{H(0)}\right)\breve{\hat{\#}}\boldsymbol{d}\,\Psi_N = 0 \quad (219)$$

$$\begin{aligned}&\sum_N \left(V_N^{E(1)\prime} + \mu\dot{I}_N^{E(1)} - \gamma_N^{E(1)}\right)\boldsymbol{d}\,\Phi_N + \\ &+ \left(V_N^{H(0)\prime}\kappa_0\, r\cos\theta + V_N^{H(1)\prime} - \mu\dot{I}_N^{H(1)} + \right. \\ &\left. + \mu\kappa_0 r\cos\theta\, \dot{I}_N^{H(0)}\right)\breve{\hat{\#}}\boldsymbol{d}\,\Psi_N = 0\end{aligned} \quad (220)$$

$$\sum_N \left(\mu\dot{\gamma}_N^{H(0)} - V_N^{H(0)}\alpha_N^2\right)\Psi_N\, r\, \boldsymbol{d}\, r\wedge\boldsymbol{d}\,\theta = 0 \quad (221)$$

$$\begin{aligned}&\sum_N \left(\mu\dot{\gamma}_N^{H(1)} - V_N^{H(1)}\alpha_N^2\right)\Psi_N\, r\,\boldsymbol{d}\, r\wedge\boldsymbol{d}\,\theta + \mu\kappa_0\dot{\gamma}_N^{H(0)}\cos\theta\,\Psi_N\, r^2\,\boldsymbol{d}\, r\wedge\boldsymbol{d}\,\theta \\ &-\kappa_0\cos\theta\, V_N^{H(0)}(\breve{\hat{\#}}\boldsymbol{d}\,\Psi_N)\wedge\boldsymbol{d}\, r + \kappa_0\, r\sin\theta\, V_N^{H(0)}(\breve{\hat{\#}}\boldsymbol{d}\,\Psi_N)\wedge\boldsymbol{d}\,\theta = 0\end{aligned} \quad (222)$$

$$\sum_N \left(I_N^{H(0)\prime} - \mu\mathcal{Y}^2\dot{V}_N^{H(0)} - \gamma_N^{H(0)}\right) \boldsymbol{d}\,\Psi_N + \left(I_N^{E(0)\prime} + \mu\mathcal{Y}^2\ \dot{V}_N^{E(0)}\right)\breve{\hat{\#}}\boldsymbol{d}\,\Phi_N = 0 \tag{223}$$

$$\begin{aligned}&\sum_N \left(I_N^{H(1)\prime} - \mu\mathcal{Y}^2\dot{V}_N^{H(1)} - \gamma_N^{H(1)}\right) \boldsymbol{d}\,\Psi_N + \\ &+ \left(I_N^{E(1)\prime} + \mu\mathcal{Y}^2\ \dot{V}_N^{E(1)} + \kappa_0\, r\cos\theta\, I_N^{E(0)\prime} - \right. \\ &\left. - \mu\mathcal{Y}^2\kappa_0 r\cos\theta\,\dot{V}_N^{E(0)}\right)\breve{\hat{\#}}\boldsymbol{d}\,\Phi_N = 0\end{aligned} \tag{224}$$

$$\sum_N \left(I_N^{E(0)}\beta_N^2 + \mu\mathcal{Y}^2\dot{\gamma}_N^{E(0)}\right)\Phi_N\, r\,\boldsymbol{d}\,r\wedge\boldsymbol{d}\,\theta = 0 \tag{225}$$

$$\begin{aligned}&\sum_N -\left(I_N^{E(1)}\beta_N^2 + \mu\mathcal{Y}^2\dot{\gamma}_N^{E(1)}\right)\Phi_N\, r\,\boldsymbol{d}\,r\wedge\boldsymbol{d}\,\theta - \\ &- \left(\mu\mathcal{Y}^2\dot{\gamma}_N^{E(0)} \quad + I_N^{E(0)}\beta_N^2\right) r^2\kappa_0\ \cos\theta\ \Phi_N\,\boldsymbol{d}\,r\wedge\boldsymbol{d}\,\theta - \\ &- I_N^{E(0)}\kappa_0\cos\theta\,(\breve{\hat{\#}}\boldsymbol{d}\,\Phi_N)\wedge\boldsymbol{d}\,r + \\ &+ I_N^{E(0)}\kappa_0\sin\theta\,(\breve{\hat{\#}}\boldsymbol{d}\,\Phi_N)\wedge r\boldsymbol{d}\,\theta = 0\end{aligned} \tag{226}$$

$$\sum_N \left(\gamma_N^{H(0)\prime} - \alpha_N^2\, I_N^{H(0)}\right) r\,\Psi_N\ \boldsymbol{d}\,r\wedge\boldsymbol{d}\,\theta\wedge\boldsymbol{d}\,z = 0 \tag{227}$$

$$\begin{aligned}
&\sum_N \Big(\gamma_N^{H(1)\prime} - \alpha_N^2\, I_N^{H(1)} + \\
&+ r\,\kappa_0 \cos\theta\, \gamma_N^{H(0)} + r I_N^{H(0)} \alpha_N^2\, \kappa_0 \cos\theta\Big)\, r\, \Psi_N\; \boldsymbol{d} r \wedge \boldsymbol{d}\theta \wedge \boldsymbol{d} z + \\
&+ \kappa_0 \cos\theta\, I_N^{H(0)} (\breve{\ddot{\#}} \boldsymbol{d}\, \Psi_N) \wedge \boldsymbol{d} r \wedge \boldsymbol{d} z - \\
&- \kappa_0 r\, \sin\theta\, I_N^{H(0)} (\breve{\ddot{\#}} \boldsymbol{d}\, \Psi_N) \wedge \boldsymbol{d}\theta \wedge \boldsymbol{d} z \\
&= 0
\end{aligned} \tag{228}$$

$$\sum_N \big(\gamma_N^{E(0)\prime} - \beta_N^2 V_N^{E(0)}\big) \mu \mathcal{Y}^2\, \Phi_N\, r\; \boldsymbol{d} r \wedge \boldsymbol{d}\theta \wedge \boldsymbol{d} z = 0 \tag{229}$$

$$\begin{aligned}
\mu\mathcal{Y}^2 \Big\{ &\sum_N \Big(\gamma_N^{E(1)\prime} - \beta_N^2 V_N^{E(1)} + \\
&+ r V_N^{E(0)} \beta_N^2 \kappa_0 \cos\theta + \\
&+ \gamma_N^{E(0)} r\, \kappa_0 \cos\theta\Big)\, \Phi_N\; r\; \boldsymbol{d} r \wedge \boldsymbol{d}\theta \wedge \boldsymbol{d} z + \\
&+ V_N^{E(0)} \kappa_0\; \cos\theta\, (\breve{\ddot{\#}} \boldsymbol{d}\, \Phi_N) \wedge \boldsymbol{d} r \wedge \boldsymbol{d} z - \\
&- V_N^{E(0)} \kappa_0\, r\; \sin\theta\, (\breve{\ddot{\#}} \boldsymbol{d}\, \Phi_N) \wedge \boldsymbol{d}\theta \wedge \boldsymbol{d} z \Big\} = 0
\end{aligned} \tag{230}$$

By integrating these equations over $\mathcal{D}$ with either Φ_M, $\breve{\ddot{\#}} \boldsymbol{d}\, \Phi_M$, Ψ_M or $\breve{\ddot{\#}} \boldsymbol{d}\, \Psi_M$ and using the mode orthogonality relations, one may project out a system that can be analysed for the perturbed amplitudes, as in the case of the cylinder with a surface perturbation. The above equations are deliberately not projected here to facilitate their modification when internal sources are included.

One may generalise this toroidal case by treating a guide with generating space-curve with arbitrary curvature $\kappa(z)$ and torsion $\mathcal{T}(z)$. This can be generalised yet further by changing to a chart adapted to a twisting guide with a variable cross-section radius given by $\rho = a\, \chi(z)$ as in the cylinder case. In the *new adapted chart* $\{r, \theta, z\}$ the appropriate orthonormal coframe now takes the form:

$$\begin{aligned} \{e^1 &= \boldsymbol{d}\,x_1 - x_2\mathcal{T}(z)\boldsymbol{d}\,z, \quad e^2 = \boldsymbol{d}\,x_2 + x_1\mathcal{T}(z)\boldsymbol{d}\,z, \\ e^3 &= (1-\kappa(z)\,x_1)\,\boldsymbol{d}\,z\} \end{aligned} \tag{231}$$

where $x_1 = r\chi(z)\cos\theta$ and $x_2 = r\chi(z)\sin\theta$.

15 Dynamic sources

In the above little discussion has been given about the structure of the current and charge densities other than the necessity that they satisfy the local conservation equation (88). It is important to realise that for applications to accelerators these sources are often strongly dynamically coupled to the fields that they produce. The interaction is fundamentally given by the Lorentz force on moving charges and the equation of motion for such charges depends non-linearly on their velocity. Since the current itself is proportional to this velocity the coupled problem is non-linear and recourse to some approximation scheme or numerical simulation is required. For sources that are produced by high speed charged particles (compared with the speed of light) the longitudinal velocity component v in a guiding structure is generally large compared with the transverse velocity so a natural first approximation is to neglect the latter. A further approximation is to assume that the longitudinal current depends on only one of the single longitudinal variables $\xi^3 \pm v\,t$ depending on the overall direction of motion of the high velocity source. Some authors then assume further that this approximate current is restricted to a line parallel to the *design orbit* in the guide. Some delicate manipulations are then required to eliminate singularities in the fields that arise from such line sources.

To construct a fully coupled system of sources and fields in the guide one must supplement the Maxwell system with an equation for the currents. A simple approach is to assume that the source is a charged relativistic fluid [7], [8], [9], [10] with $4-$velocity V and $4-$acceleration $\mathcal{A}$ satisfying the equation of motion

$$m\,\mathcal{A} = -q\,i_V F \tag{232}$$

where m and q denote the mass and charge of the elementary constituents of the fluid respectively. In terms of the Levi-Civita covariant derivative

∇, $\mathcal{A} \equiv \nabla_V \tilde{V}$ and V is a unit time-like future pointing vector field on spacetime:

$$g(V,V) = -1 \tag{233}$$

with the $1-$ form $\tilde{V} = g(V,-)$. The sources are then the components of the the $4-$current $3-$form

$$\underset{(3)}{j} = \rho_V \star \tilde{V} \tag{234}$$

and ρ_V is the *proper* charge density of the source.

It is convenient to recast (232) in terms of the exterior derivative on spacetime

$$i_V\big(d\tilde{V} + \frac{q}{m}F\big) = 0 \tag{235}$$

since one can now dimensionally reduce this equation in a similar manner to that applied to the Maxwell system. In terms of the guide frame sources $\underset{(2)}{J}(\xi)$ and $\underset{(0)}{\rho}(\xi)$

$$\underset{(3)}{j}(\xi) = -\underset{(2)}{J}(\xi) \wedge d\,t + \underset{(0)}{\rho}(\xi)\#1 \tag{236}$$

where the 2+1 split with respect to $\boldsymbol{d}\xi^3$ gives

$$\underset{(2)}{J} = \underset{(1)}{\hat{J}} \wedge \boldsymbol{d}\xi^3 + \underset{(0)}{\hat{J}}\hat{\#}1 \tag{237}$$

Thus the proper charge density is given in terms of the guide frame sources since

$$\rho_V^2 = -\star\big(\underset{(3)}{j} \wedge \star \underset{(3)}{j}\big) \tag{238}$$

and the $4-$ velocity follows from (234). It remains to supplement (232) with boundary conditions for the independent components of V. This is a delicate issue since the effects of the guide boundary on the sources can be very complex.

One approach is to base a perturbative scheme on an ultra-relativistic zeroth-order approximation $\underset{(0)}{\rho^{(0)}}(\xi)$, $\underset{(0)}{\hat{J}^{(0)}}(\xi)$, $\underset{(1)}{\hat{J}^{(1)}}(\xi)$ in which the sources are prescribed, compatible with (88) and specified external (accelerating) fields. The boundary conditions on the higher order perturbations (generating higher order wake fields) are then prescribed in terms of source components associated with the Dirichlet and Neumann electromagnetic modes excited in the guide.

To effect this methodology one first splits (235) with respect to $\boldsymbol{d}\,\xi^3$ and adopts the following source expansions:

$$\underset{(0)}{\rho}(\xi) = \underset{(0)}{\rho^{(0)}}(\xi) + \sum_N \epsilon\, \underset{(0)}{\rho^E_N}(\epsilon, t, \xi^3)\Phi_N(\xi^1, \xi^2) + \sum_M \epsilon\, \underset{(0)}{\rho^H_M}(\epsilon, t, \xi^3)\Psi_M(\xi^1, \xi^2) \tag{239}$$

$$\begin{aligned} \underset{(0)}{\hat{J}}(\xi) &= \underset{(0)}{\hat{J}^{(0)}}(\xi) + \sum_N \epsilon\, \underset{(0)}{\gamma^{E_J}_N}(\epsilon, t, \xi^3)\Phi_N(\xi^1, \xi^2) + \\ & \sum_M \epsilon\, \underset{(0)}{\gamma^{H_J}_M}(\epsilon, t, \xi^3)\Psi_M(\xi^1, \xi^2) \end{aligned} \tag{240}$$

$$\begin{aligned} \underset{(1)}{\hat{J}}(\xi) &= \underset{(1)}{\hat{J}^{(0)}}(\xi) + \sum_N \epsilon\, \underset{(0)}{V^{E_J}_N}(\epsilon, t, \xi^3)\boldsymbol{d}\,\Phi_N(\xi^1, \xi^2) + \\ & \sum_N \epsilon\, \underset{(0)}{V^{H_J}_N}(\epsilon, t, \xi^3)\hat{\#}\,\boldsymbol{d}\,\Phi_N(\xi^1, \xi^2) \\ & + \sum_M \epsilon\, \underset{(0)}{I^{H_J}_M}(\epsilon, t, \xi^3)\hat{\#}\boldsymbol{d}\,\Psi_M(\xi^1, \xi^2) + \\ & \sum_M \epsilon\, \underset{(0)}{I^{E_J}_M}(\epsilon, t, \xi^3)\boldsymbol{d}\,\Psi_M(\xi^1, \xi^2) \end{aligned} \tag{241}$$

The source amplitudes $\underset{(0)}{\rho^E_N}(\epsilon, t, \xi^3)$, $\underset{(0)}{\rho^H_N}(\epsilon, t, \xi^3)$, $\underset{(0)}{\gamma^{E_J}_N}(\epsilon, t, \xi^3)$, $\underset{(0)}{\gamma^{H_J}_N}(\epsilon, t, \xi^3)$, $\underset{(0)}{V^{E_J}_N}(\epsilon, t, \xi^3)$, $\underset{(0)}{V^{H_J}_N}(\epsilon, t, \xi^3)$, $\underset{(0)}{I^{H_J}_M}(\epsilon, t, \xi^3)$, $\underset{(0)}{I^{E_J}_M}(\epsilon, t, \xi^3)$ are taken analytic in ϵ and the mode summations follow the ranges specified by the spectral content as indicated in the earlier sections. Thus by a Taylor expansion of these amplitudes about $\epsilon = 0$ one may project the fully coupled system consisting of the Maxwell equations and (235) into a perturbative scheme based on the zeroth-order sources and the prescribed externally applied electromagnetic fields. The results of this strategy for different guide geometries and bunched source configurations will be presented elsewhere.

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