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STOCHASTIC PROPERTIES OF THE FROBENIUS-PERRON OPERATOR

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Abstract

In the present paper the Renormalization Group (RG) method is adopted as a tool for a constructive analysis of the properties of the Frobenius-Perron Operator. The renormalization group reduction of a generic symplectic map in the case, where the unperturbed rotation frequency of the map is far from structural resonances driven by the kick perturbation has been performed in detail. It is further shown that if the unperturbed rotation frequency is close to a resonance, the reduced RG map of the Frobenius-Perron operator (or phase-space density propagator) is equivalent to a discrete Fokker-Planck equation for the renormalized distribution function. The RG method has been also applied to study the stochastic properties of the standard Chirikov-Taylor map.

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§1. Introduction

Recursive maps represent a useful and powerful tool to model and to facilitate the understanding of the physical processes taking place in complex nonlinear systems. In particular, they are widely used to study the various transition scenarios from regular to chaotic behaviour in nonlinear dynamical systems,[1)–3)] to simulate physical systems exhibiting anomalous diffusion,[4)] or to analyze the underlying dynamics in time series with $1/f$ noise in their power spectrum.[5),6)] Iterative maps provide a convenient and effective method to investigate single-particle dynamics in accelerators and storage rings.[7),8)]

The extremely complicated behaviour of specific trajectories in chaotic systems strongly suggests a probabilistic approach to the dynamics. Instead of tracing an individual trajectory in phase space, one employs a statistical mechanics approach by means of a distribution function of an ensemble of trajectories. The Frobenius-Perron operator of a phase-space density (distribution) function, which sometimes is called the Transfer Operator of that function or a phase-space density propagator, provides a tool for studying the dynamics of the iteration of the distribution function itself. The iterative map yields complete information of how the value of an individual phase-space point jumps around during successive iterations, so that one gets a good sense of the point dynamics but no sense of how iteration acts on densities with support on sets in phase space. The latter gap is filled by the Frobenius-Perron operator, which provides a rule to determine how the evolution of densities over repeated iterations is accomplished.

In the present paper we adopt the Renormalization Group (RG) method for a constructive analysis of the properties of the Frobenius-Perron Operator. The basic idea of the method is to absorb secular or divergent terms of the naive perturbation solution into renormalized integration constants (amplitudes). As a result one obtains an evolution law embedded in an evolution equation the renormalized amplitudes must satisfy, which describes the large-scale dynamics of the system. The latter is usually called the RG equation. The stages in the renormalization group reduction of a particular physical system are quite general and well defined, which makes the RG method universal and independent on the concrete details of the underlying dynamics and physical processes involved.[7),9)–12)]

The paper is organized as follows. The next paragraph serves as a reminder of the basic definition, derivation and properties of the Frobenius-Perron operator and provides the starting point for the subsequent exposition. In Paragraph 3, we work out in detail the renormalization group reduction of a generic symplectic map in the case, where the unperturbed rotation frequency of the map is far from structural resonances driven by the kick perturbation. Paragraph 4 deals with the resonance structure of a symplectic map. It is shown that in the case, where the unperturbed rotation frequency is close to a resonance, the reduced RG map of the Frobenius-Perron operator is equivalent to a discrete Fokker-Planck equation for the renormalized distribution function. In Paragraph 5 we apply the RG method to study the stochastic properties of the standard Chirikov-Taylor map, and (re)derive the diffusion coefficient in the quasi-linear approximation. In Paragraph 6 a few examples concerning the application of the results thus obtained are worked out in detail. Finally, Paragraph 7 is dedicated to conclusions and outlook.

§2. The Frobenius-Perron Operator for the Henon Map

The Henon map is defined by the following expression:[7)]

$$\mathbf{z}_{n+1} = \begin{pmatrix} x_{n+1} \\ p_{n+1} \end{pmatrix} = \mathcal{R}_\omega \begin{pmatrix} x_n \\ p_n - \mathcal{S} x_n^2 \end{pmatrix}, \tag{2·1}$$

where

$$\mathcal{R}_\omega = \begin{pmatrix} \cos\omega & \sin\omega \\ -\sin\omega & \cos\omega \end{pmatrix}, \tag{2·2}$$

is the rotation matrix for one period of the map. In terms of accelerator physics application this is equivalent to one revolution along the accelerator lattice. The frequency ω and the parameter $\mathcal{S}$

$$\omega = 2\pi\nu, \qquad\qquad \mathcal{S} = \frac{l\lambda_0(\theta_0)\beta^{3/2}(\theta_0)}{2R^3}, \tag{2·3}$$

are related to the unperturbed betatron tune ν and to the strength of the sextupole (cubic nonlinearity) perturbation λ_0. Here l is the length of the sextupole, θ_0 is its location on the azimuth of the machine and R is the mean radius.

The Henon map can be written as

$$\mathbf{Z}_{n+1} = \mathcal{R}_\omega^T \mathbf{z}_{n+1} = \begin{pmatrix} x_n \\ p_n - \mathcal{S} x_n^2 \end{pmatrix}, \tag{2·4}$$

where $\mathcal{R}_\omega^T$ denotes the transposed of the matrix (2·2). The Frobenius-Perron operator[7)] can be calculated explicitly. We have:

$$f_{n+1}(x,p) = \widehat{\mathbf{U}} f_n(x,p) = \int \mathrm{d}\xi \mathrm{d}\eta \delta(X-\xi)\delta\big(P-\eta+\mathcal{S}\xi^2\big) f_n(\xi,\eta)$$
$$= f_n\big(X, P+\mathcal{S}X^2\big). \tag{2·5}$$

Introducing the formal small parameter ϵ and the action-angle variables

$$x = \sqrt{2J}\cos a, \qquad\qquad p = -\sqrt{2J}\sin a, \tag{2·6}$$

with

$$J = \frac{1}{2}\big(x^2 + p^2\big), \qquad\qquad a = -\arctan\left(\frac{p}{x}\right), \tag{2·7}$$

we write the Frobenius-Perron operator represented by equation (2·5) in the form

$$f_{n+1}(a + \omega, J) = f_n\big(x, p + \epsilon \mathcal{S} x^2\big). \tag{2·8}$$

§3. Renormalization Group Treatment of the Frobenius-Perron Operator

The generalization of the Frobenius-Perron operator (2·8) for a generic symplectic map with rotation is straightforward. We have

$$f_{n+1}(a + \omega, J) = f_n(x, p + \epsilon \partial_x V_N), \tag{3·1}$$

where $V_N(x)$ is a potential and ∂_x denotes partial differentiation with respect to x. In the case of the Henon map for instance, the potential has the form

$$V_N(x) = \frac{\mathcal{S} x^3}{3}. \tag{3·2}$$

Equation (3·1) can be written as

$$f_{n+1}(a + \omega, J) = e^{\epsilon (\partial_x V_N) \partial_p} f_n(a, J). \tag{3·3}$$

Since the potential V_N does not depend on the momentum variable p,

$$\widehat{\mathbf{L}}_V = (\partial_x V_N)\partial_p - (\partial_p V_N)\partial_x = (\partial_x V_N)\partial_p, \tag{3·4}$$

where $\widehat{\mathbf{L}}_V$ is the Liouvillean operator associated with V_N. Therefore, equation (3·3) becomes

$$f_{n+1}(a + \omega, J) = e^{\epsilon \widehat{\mathbf{L}}_V} f_n(a, J). \tag{3·5}$$

We assume that the potential V_N, written in action-angle variables can be split as follows

$$V_N(a, J) = V_0(J) + V(a, J). \tag{3·6}$$

Respectively, the Liouvillean operator can be written as

$$\widehat{\mathbf{L}}_V = \widehat{\mathbf{L}}_0 + \widehat{\mathbf{L}}, \tag{3·7}$$

where

$$\widehat{\mathbf{L}}_0 = -\omega_V(J)\partial_a, \qquad\qquad \widehat{\mathbf{L}} = (\partial_a V)\partial_J - (\partial_J V)\partial_a, \tag{3·8}$$

and

$$\omega_V(J) = \frac{\partial V_0}{\partial J}. \tag{3·9}$$

First of all, we consider the case, where the rotation frequency ω is away from non-linear resonances driven by the potential V. Following the standard procedure of the RG method,[7),9)] we seek a solution to equation (3·5) by naive perturbation expansion

$$f_n(a, J) = \sum_{k=0}^{\infty} \epsilon^k f_n^{(k)}(a, J), \tag{3·10}$$

where the unknown functions $f_n^{(k)}(a, J)$ have to be determined order by order. The zero-order equation

$$f_{n+1}^{(0)}(a + \omega, J) = f_n^{(0)}(a, J), \tag{3·11}$$

has the obvious solution

$$f_n^{(0)}(a, J) = e^{-n\omega\partial_a} F(a, J) = F(a - n\omega, J). \tag{3·12}$$

To this end $F(a, J)$ is an arbitrary function of its arguments, and will be the subject of the renormalization group reduction in the sequel.

The first-order equation can be written as follows

$$f_{n+1}^{(1)}(a + \omega, J) - f_n^{(1)}(a, J) = \widehat{\mathbf{L}}_V F(a - n\omega, J). \tag{3·13}$$

Standard but cumbersome algebra yields the solution to equation (3·13) in the form

$$f_n^{(1)}(a, J) = \left(n\widehat{\mathbf{L}}_0 + \widehat{\mathcal{L}}_\omega\right) F(a - n\omega, J), \tag{3·14}$$

where

$$\widehat{\mathcal{L}}_\omega = (\partial_a V_\omega)\partial_J - (\partial_J V_\omega)\partial_a. \tag{3·15}$$

Furthermore, the potential $V_\omega(a, J)$ is defined according to the expression

$$V_\omega(a, J) = V_1\left(a - \frac{\omega}{2}, J\right), \qquad V_1(a, J) = \sum_{m \neq 0} \frac{V_m(J) e^{ima}}{2i \sin(m\omega/2)}. \tag{3·16}$$

Some of the details of the calculation can be found in Appendix A.

The second order equation is

$$f_{n+1}^{(2)}(a + \omega, J) - f_n^{(2)}(a, J) = \widehat{\mathbf{L}}_V f_n^{(1)}(a, J) + \frac{\widehat{\mathbf{L}}_V^2}{2} F(a - n\omega, J). \tag{3·17}$$

Since we are interested in the secular solution of equation (3·17), we retain on its right-hand-side only terms that would yield a secular contribution. Thus, the second order equation giving rise to secular solution can be written as

$$f_{n+1}^{(2)}(a + \omega, J) - f_n^{(2)}(a, J) = \left[\left(n + \frac{1}{2}\right)\widehat{\mathbf{L}}_0^2 + n\widehat{\mathbf{L}}\widehat{\mathbf{L}}_0 + \Omega(\omega, J)\partial_a\right] F(a - n\omega, J), \tag{3·18}$$

where

$$\Omega(\omega, J) = \sum_{m=1}^{\infty} m \cot\left(\frac{m\omega}{2}\right) \partial_J (V_m \partial_J V_m). \tag{3·19}$$

Omitting the details of the calculation (presented in Appendix A), we can write the second-order solution as

$$f_n^{(2)}(a, J) = \left[\frac{n^2}{2}\widehat{\mathbf{L}}_0^2 + n\widehat{\mathcal{L}}_\omega \widehat{\mathbf{L}}_0 + n\Omega(\omega, J)\partial_a\right] F(a - n\omega, J) + \text{non secular terms}. \tag{3·20}$$

To remove secular terms (proportional to n and n^2) in the first-order (3·14) and the second-order solution (3·20), we define a renormalization group transformation $F(a, J) \to \widetilde{F}(a, J; n)$ by collecting all terms proportional to $F(a - n\omega, J)$

$$\widetilde{F}(a - n\omega, J; n) = \left[1 + \epsilon n \widehat{\mathbf{L}}_0 + \epsilon^2 \left(\frac{n^2}{2}\widehat{\mathbf{L}}_0^2 + n\Omega\partial_a\right)\right] F(a - n\omega, J). \tag{3·21}$$

Solving perturbatively equation (3·21) for $F(a - n\omega, J)$ in terms of $\widetilde{F}(a - n\omega, J; n)$, we obtain

$$F(a - n\omega, J) = \left(1 - \epsilon n \widehat{\mathbf{L}}_0 + \dots\right) \widetilde{F}(a - n\omega, J; n). \tag{3·22}$$

Following Reference,[7),12)] we define a discrete version of the RG equation by considering the difference

$$\begin{aligned} &\widetilde{F}(a - n\omega, J; n+1) - \widetilde{F}(a - n\omega, J; n) \\ &= \left\{\epsilon \widehat{\mathbf{L}}_0 + \epsilon^2 \left[\left(n + \frac{1}{2}\right)\widehat{\mathbf{L}}_0^2 + \Omega\partial_a\right]\right\} F(a - n\omega, J). \end{aligned} \tag{3·23}$$

Substituting the expression for $F(a - n\omega, J)$ in terms of $\widetilde{F}(a - n\omega, J; n)$ from equation (3·22), we can eliminate secular terms up to $O(\epsilon^2)$. The result is

$$\widetilde{F}(a - n\omega, J; n+1) - \widetilde{F}(a - n\omega, J; n) = \left[\epsilon \widehat{\mathbf{L}}_0 + \epsilon^2 \left(\frac{\widehat{\mathbf{L}}_0^2}{2} + \Omega\partial_a\right)\right] \widetilde{F}(a - n\omega, J; n). \tag{3·24}$$

Equation (3·24) is the RG equation. It describes the evolution of the distribution function on slow time scales in addition to the fast oscillations with a fundamental frequency ω.

An important remark is in order at this point. Note that once the renormalization transformation has been performed, the second term in the second-order solution (3·20) is eliminated as well. Combining it with the second (non secular) term in the first-order solution (3·14), we obtain

$$\epsilon \widehat{\mathcal{L}}_\omega F + \epsilon^2 n \widehat{\mathcal{L}}_\omega \widehat{\mathbf{L}}_0 F = \epsilon \widehat{\mathcal{L}}_\omega \left(1 - \epsilon n \widehat{\mathbf{L}}_0\right) \widetilde{F}(n) + \epsilon^2 n \widehat{\mathcal{L}}_\omega \widehat{\mathbf{L}}_0 \widetilde{F}(n) = \epsilon \widehat{\mathcal{L}}_\omega \widetilde{F}(n). \tag{3·25}$$

To first order in the perturbation parameter ϵ the renormalized solution to equation (3·5) can be written as

$$f_n(a,J) = \left(1+\epsilon\widehat{\mathcal{L}}_\omega\right)\widetilde{F}(a-n\omega,J;n), \tag{3·26}$$

where the renormalized "amplitude" $\widetilde{F}(a-n\omega,J;n)$ satisfies the RG equation (3·24). In the continuous limit equation (3·24) acquires the form

$$\frac{\partial\widetilde{F}(a-n\omega,J;n)}{\partial n} = \left[\epsilon\widehat{\mathbf{L}}_0 + \epsilon^2\left(\frac{\widehat{\mathbf{L}}_0^2}{2} + \Omega\partial_a\right)\right]\widetilde{F}(a-n\omega,J;n). \tag{3·27}$$

Provided $\widehat{\mathbf{L}}_0 \neq 0$ (in the case, where the potential V_N is not antisymmetric) the latter is a Fokker-Planck equation with the Fokker-Planck operator acting on the angle variable only. A relevant example of a cubic map will be considered in Paragraph 6.

§4. Resonance Structure of a Symplectic Map

The solution (3·14) to the first-order perturbation equation (3·13) was obtained under the assumption that the unperturbed betatron tune ν is sufficiently far from any structural nonlinear resonance of the form $m_0\nu = 1$, where m_0 is an integer. In the present paragraph, we assume that

$$\omega = \omega_0 + \epsilon\delta_1 + \epsilon^2\delta_2 + \dots, \qquad \omega_0 = \frac{2\pi}{m_0}. \tag{4·1}$$

Moreover, for the sake of simplicity, we assume that there are no higher angle-dependent harmonics in the Fourier spectrum of $V(a,J)$ that would drive higher-order resonances of the form $pm_0\nu = p$, where p is an integer. However, results can be generalized easily to take into account this case as well.

Proceeding as in Paragraph 3, we write the zero-order solution as

$$f_n^{(0)}(a,J) = e^{-n\omega_0\partial_a}F(a,J) = F(a-n\omega_0,J). \tag{4·2}$$

The first-order equation in the resonant case can be written as follows

$$f_{n+1}^{(1)}(a+\omega_0,J) - f_n^{(1)}(a,J) = \left(\widehat{\mathbf{L}}_1 + \widehat{\mathbf{L}}\right)F(a-n\omega_0,J), \tag{4·3}$$

where

$$\widehat{\mathbf{L}}_1 = -\delta_1\partial_a + \widehat{\mathbf{L}}_0. \tag{4·4}$$

The solution to equation (4·3) is readily found in a standard manner, analogous to that already used in Paragraph 3 (see also Appendix A). The result is

$$f_n^{(1)}(a,J) = \left[n\left(\widehat{\mathbf{L}}_1 + \widehat{\mathbf{L}}_R\right) + \widehat{\mathcal{L}}'_\omega\right]F(a-n\omega_0,J), \tag{4·5}$$

where

$$\widehat{\mathbf{L}}_R = (\partial_a V_R)\partial_J - (\partial_J V_R)\partial_a, \tag{4·6}$$

is the resonant Liouvillean operator and

$$V_R(a,J) = \sum_{m=\pm m_0} V_m(J) e^{ima} = 2V_{m0}(J)\cos m_0 a, \tag{4·7}$$

is the resonant potential. Furthermore,

$$\widehat{\mathcal{L}}'_\omega = (\partial_a V'_\omega)\partial_J - (\partial_J V'_\omega)\partial_a, \tag{4·8}$$

where now the potential $V'_\omega(a,J)$ is defined according to the expression

$$V'_\omega(a,J) = V'_1\Big(a - \frac{\omega_0}{2}, J\Big), \qquad V'_1(a,J) = \sum_{m\neq\pm m_0} \frac{V_m(J)e^{ima}}{2i\sin(m\omega_0/2)}. \tag{4·9}$$

The second-order equation in the resonant case can be written as

$$f^{(2)}_{n+1}(a+\omega_0,J) - f^{(2)}_n(a,J) = -\delta_1 \partial_a f^{(1)}_{n+1}(a+\omega_0,J)$$

$$-\frac{1}{2}\big(\delta_1^2\partial^2_{aa} + 2\delta_2\partial_a\big) f^{(0)}_{n+1}(a+\omega_0,J) + \widehat{\mathbf{L}}_V f^{(1)}_n(a,J) + \frac{\widehat{\mathbf{L}}_V^2}{2} F(a-n\omega_0,J). \tag{4·10}$$

Similar to Paragraph 3, we again retain terms on the right-hand-side of equation (4·10) that would yield secular contributions to the second-order solution. Thus the second-order solution can be written as

$$f^{(2)}_n(a,J) = n\bigg\{(\Omega' - \delta_2)\partial_a + \frac{\delta_1}{2}\Big[\widehat{\mathbf{L}}_R, \partial_a\Big]\bigg\} F(a-n\omega_0,J)$$

$$+\frac{n^2}{2}\Big(\widehat{\mathbf{L}}_1 + \widehat{\mathbf{L}}_R\Big)^2 F(a-n\omega_0,J) + n\widehat{\mathcal{L}}'_\omega \Big(\widehat{\mathbf{L}}_1 + \widehat{\mathbf{L}}_R\Big) F(a-n\omega_0,J) + \text{non secular terms}, \tag{4·11}$$

where

$$\Big[\widehat{\mathbf{L}}_R, \partial_a\Big] = \widehat{\mathbf{L}}_R \partial_a - \partial_a \widehat{\mathbf{L}}_R, \tag{4·12}$$

is the commutator of the operators $\widehat{\mathbf{L}}_R$ and ∂_a, and

$$\Omega'(\omega_0,J) = \sum_{m=1}^{\infty}{}' \, m\cot\Big(\frac{m\omega_0}{2}\Big)\partial_J(V_m \partial_J V_m). \tag{4·13}$$

is the new nonlinear tune shift. Here the prime in the above sum implies that the term with $m = m_0$ is excluded from the sum.

Repeating the steps that brought us along from equation (3·21) to equation (3·24) in the previous paragraph, we obtain the RG equation in the resonant case

$$\widetilde{F}(a_n, J; n+1) - \widetilde{F}(a_n, J; n)$$

$$= \left\{ \epsilon \left(\widehat{\mathbf{L}}_1 + \widehat{\mathbf{L}}_R \right) + \epsilon^2 \left[(\Omega' - \delta_2)\partial_a + \frac{\delta_1}{2} \left[\widehat{\mathbf{L}}_R, \partial_a \right] + \frac{1}{2} \left(\widehat{\mathbf{L}}_1 + \widehat{\mathbf{L}}_R \right)^2 \right] \right\} \widetilde{F}(a_n, J; n), \qquad (4{\cdot}14)$$

where $a_n = a - n\omega_0$. Analogously to Paragraph 3, to first order in the perturbation parameter ϵ the renormalized solution of equation (3·5) in the resonant case is represented by the expression

$$f_n(a, J) = \left(1 + \epsilon \widehat{\mathcal{L}}'_\omega \right) \widetilde{F}(a - n\omega_0, J; n), \qquad (4{\cdot}15)$$

As in the non resonant case, the renormalization group transformation ensures the elimination of the last term on the right-hand-side of equation (4·11) [combined with the last term on the right-hand-side of equation (4·5)] up to $O(\epsilon^2)$. In the continuous limit the RG equation (4·14) is a Fokker-Planck equation with a Fokker-Planck operator acting on both action and angle variables.

§5. Stochastic Properties of the Standard Map

The present paragraph is dedicated to the analysis of the stochastic properties of the standard map. The Frobenius-Perron operator can be calculated explicitly and the Frobenius-Perron equation can be written as[7)]

$$f_{n+1}(a, J) = e^{-J\partial_a} e^{\epsilon \sin a \partial_J} f_n(a, J), \qquad (5{\cdot}1)$$

or

$$e^{J\partial_a} f_{n+1}(a, J) = e^{\epsilon \sin a \partial_J} f_n(a, J). \qquad (5{\cdot}2)$$

We introduce a new variable ξ according to the equation

$$\xi = \frac{J - J_0}{\sqrt{\epsilon}}, \qquad J = J_0 + \sqrt{\epsilon}\xi, \qquad (5{\cdot}3)$$

where J_0 is a fixed value of the action variable. The first case we will consider is the one, where $J_0 \neq 0$. The Frobenius-Perron equation (5·2) can be rewritten as

$$e^{\left(J_0 + \sqrt{\epsilon}\xi\right)\partial_a} f_{n+1}(a, \xi) = e^{\sqrt{\epsilon} \sin a \partial_\xi} f_n(a, \xi). \qquad (5{\cdot}4)$$

The zero-order equation

$$e^{J_0\partial_a} f_{n+1}^{(0)}(a, \xi) = f_{n+1}^{(0)}(a + J_0, \xi) = f_n^{(0)}(a, \xi), \qquad (5{\cdot}5)$$

has the obvious solution

$$f_n^{(0)}(a, \xi) = e^{-nJ_0\partial_a} F(a, \xi) = F(a - nJ_0, \xi). \qquad (5{\cdot}6)$$

The first-order perturbation equation can be written as follows

$$e^{J_0 \partial_a} f_{n+1}^{(1)}(a, \xi) - f_n^{(1)}(a, \xi) = (-\xi \partial_a + \sin a \partial_\xi) F(a - nJ_0, \xi). \tag{5·7}$$

The latter can be solved in a straightforward manner and the result is

$$f_n^{(1)}(a, \xi) = -\left[n\xi \partial_a + \frac{\cos(a - J_0/2)}{2 \sin(J_0/2)} \partial_\xi \right] F(a - nJ_0, \xi). \tag{5·8}$$

Proceeding further, we write the second-order perturbation equation

$$e^{J_0 \partial_a} f_{n+1}^{(2)} - f_n^{(2)} = -e^{J_0 \partial_a} \left(\xi \partial_a f_{n+1}^{(1)} + \frac{\xi^2}{2} \partial_{aa}^2 f_{n+1}^{(0)} \right) + \sin a \partial_\xi f_n^{(1)} + \frac{1}{2} \sin^2 a \partial_{\xi\xi}^2 f_n^{(0)}. \tag{5·9}$$

Retaining terms on the right-hand-side of equation (5·9) that would give rise to secular contribution, we cast the latter in an explicit form according to the expression

$$e^{J_0 \partial_a} f_{n+1}^{(2)} - f_n^{(2)} = \xi^2 \left(n + \frac{1}{2} \right) \partial_{aa}^2 F(a - nJ_0, \xi) - n \sin a \partial_\xi \xi \partial_a F(a - nJ_0, \xi)$$
$$- \frac{\cot(J_0/2)}{4} \sin 2a \partial_{\xi\xi}^2 F(a - nJ_0, \xi). \tag{5·10}$$

The secular solution of equation (5·10) is

$$f_n^{(2)}(a, \xi) = \frac{n^2}{2} \xi^2 \partial_{aa}^2 F(a - nJ_0, \xi) + n \frac{\cos(a - J_0/2)}{2 \sin(J_0/2)} \partial_\xi \xi \partial_a F(a - nJ_0, \xi). \tag{5·11}$$

Following out a renormalization procedure similar to the one performed in Paragraphs 3 and 4, we obtain the RG equation

$$\widetilde{F}(a - nJ_0, \xi; n+1) - \widetilde{F}(a - nJ_0, \xi; n) = \left(-\sqrt{\epsilon} \xi \partial_a + \frac{\epsilon \xi^2}{2} \partial_{aa}^2 \right) \widetilde{F}(a - nJ_0, \xi; n). \tag{5·12}$$

Note that in the case of $J_0 = \pi$, the third term on the right-hand-side of equation (5·10) vanishes and we are again left with the above expressions for the secular second-order solution (5·11) and for the renormalization group map (5·12) (taken for $J_0 = \pi$).

To complete the present paragraph, we consider the case of $J_0 = 0$. Equation (5·3) implies a simple (non canonical) scaling of the canonical variables, assuming that J is a slow variable. Then, equation (5·1) can be rewritten as

$$f_{n+1}(a, \xi) = e^{-\sqrt{\epsilon} \xi \partial_a} e^{\sqrt{\epsilon} \sin a \partial_\xi} f_n(a, \xi). \tag{5·13}$$

Note that now the perturbation parameter is $\sqrt{\epsilon}$ rather than ϵ. Using the Campbell-Baker-Hausdorff identity

$$\exp\left(\alpha \widehat{\mathbf{A}}\right) \exp\left(\beta \widehat{\mathbf{B}}\right) = \exp\left(\alpha \widehat{\mathbf{A}} + \beta \widehat{\mathbf{B}} + \frac{\alpha\beta}{2} \left[\widehat{\mathbf{A}}, \widehat{\mathbf{B}}\right] + \dots\right), \tag{5·14}$$

for any operators $\widehat{\mathbf{A}}$ and $\widehat{\mathbf{B}}$ and any parameters α and β, we cast equation (5·13) in the form

$$f_{n+1}(a,\xi) = \exp\left(\sqrt{\epsilon}\widehat{\mathbf{L}}_S + \frac{\epsilon}{2}\widehat{\mathbf{C}} + \dots\right) f_n(a,\xi). \tag{5·15}$$

Here

$$\widehat{\mathbf{L}}_S = -\xi\partial_a + \sin a\partial_\xi, \qquad \widehat{\mathbf{C}} = [\sin a\partial_\xi, \xi\partial_a] = \sin a\partial_a - \xi\cos a\partial_\xi. \tag{5·16}$$

Next we proceed with solving of equation (5·15) order by order.

The zero-order solution is trivial to find:

$$f_n^{(0)}(a,\xi) = F(a,\xi), \tag{5·17}$$

where as before, $F(a,\xi)$ is a generic function of its arguments, playing the role of an integration constant. The first-order equation

$$f_{n+1}^{(1)}(a,\xi) - f_n^{(1)}(a,\xi) = \widehat{\mathbf{L}}_S F(a,\xi), \tag{5·18}$$

has the obvious solution

$$f_n^{(1)}(a,\xi) = n\widehat{\mathbf{L}}_S F(a,\xi). \tag{5·19}$$

The second-order equation can be written explicitly as

$$f_{n+1}^{(2)}(a,\xi) - f_n^{(2)}(a,\xi) = \left[\left(n + \frac{1}{2}\right)\widehat{\mathbf{L}}_S^2 + \frac{\widehat{\mathbf{C}}}{2}\right] F(a,\xi). \tag{5·20}$$

Its solution is readily found to be

$$f_n^{(2)}(a,\xi) = \frac{1}{2}\left(n^2\widehat{\mathbf{L}}_S^2 + n\widehat{\mathbf{C}}\right) F(a,\xi). \tag{5·21}$$

Performing the already familiar renormalization transformation in analogy to what has been done in the previous paragraphs, we obtain the RG equation

$$\widetilde{F}(a,\xi;n+1) - \widetilde{F}(a,\xi;n) = \left[\sqrt{\epsilon}\widehat{\mathbf{L}}_S + \frac{\epsilon}{2}\left(\widehat{\mathbf{L}}_S^2 + \widehat{\mathbf{C}}\right)\right]\widetilde{F}(a,\xi;n). \tag{5·22}$$

Averaging the RG equation (5·22) over the angle variable a, we obtain a discrete version of the diffusion equation

$$\widetilde{F}_0(\xi;n+1) - \widetilde{F}_0(\xi;n) = \frac{\epsilon}{4}\partial_{\xi\xi}^2\widetilde{F}_0(\xi;n) = \frac{\epsilon^2}{4}\partial_{JJ}^2\widetilde{F}_0(\xi;n), \tag{5·23}$$

for the angle-independent part of the distribution function $\widetilde{F}_0(\xi;n)$ in quasi-linear approximation. Higher order contributions to the diffusion coefficient $\epsilon^2/2$ can be obtained by direct analysis of the Frobenius-Perron operator.[7)]

§6. Examples

Let us now consider a few examples. In the non resonant case of the Henon map $\left(\widehat{\mathbf{L}}_0 \equiv 0\right)$ equation (3·27) can be written in the form

$$\frac{\partial \widetilde{F}(a - n\omega, J; n)}{\partial n} = \Omega_H(a, J)\partial_a \widetilde{F}(a - n\omega, J; n), \tag{6·1}$$

where

$$\Omega_H(\omega, J) = \frac{\mathcal{S}^2 J}{8}\left[3\cot\left(\frac{\omega}{2}\right) + \cot\left(\frac{3\omega}{2}\right)\right]. \tag{6·2}$$

Equation (6·1) written in alternative form

$$\frac{\partial \widetilde{F}(a, J; n)}{\partial n} = [-\omega + \Omega_H(a, J)]\partial_a \widetilde{F}(a, J; n), \tag{6·3}$$

describes regular motion with a frequency $\omega - \Omega_H$, and effective Hamiltonian

$$H_{eff}(J) = \omega J - \frac{\mathcal{S}^2}{16}\left[3\cot\left(\frac{\omega}{2}\right) + \cot\left(\frac{3\omega}{2}\right)\right] J^2. \tag{6·4}$$

The next example we would like to consider is the case of a cubic map with a potential of the form

$$V_N(x) = \frac{\mathcal{C}x^4}{4}. \tag{6·5}$$

Taking into account equation (3·6), we also have

$$V_0(J) = \frac{3\mathcal{C}J^2}{8}, \qquad V(a, J) = \frac{\mathcal{C}J^2}{2}\cos 2a + \frac{\mathcal{C}J^2}{8}\cos 4a. \tag{6·6}$$

Furthermore, the nonlinear tune shift can be expressed according to

$$\omega_V(J) = \frac{3\mathcal{C}J}{4}, \qquad \Omega_C(\omega, J) = \frac{3\mathcal{C}^2 J^2}{32}(8\cot\omega + \cot 2\omega). \tag{6·7}$$

Thus, equation (3·27) can be written as

$$\frac{\partial \widetilde{F}(a, J; n)}{\partial n} = -\widetilde{\omega}\partial_a \widetilde{F}(a, J; n) + \frac{\omega_V^2}{2}\partial_{aa}^2 \widetilde{F}(a, J; n), \tag{6·8}$$

where

$$\widetilde{\omega}(\omega, J) = \omega + \omega_V - \frac{3\mathcal{C}^2 J^2}{32}(8\cot\omega + \cot 2\omega). \tag{6·9}$$

Equation (6·8) can be readily solved, yielding the result

$$\widetilde{F}(a, J; n) = \sum_k \widetilde{F}_k(J; 0) e^{ik(a - n\widetilde{\omega})} e^{-k^2 \omega_V^2 n/2}. \tag{6·10}$$

The latter indicates that the renormalized distribution function $\widetilde{F}(a, J; n)$ rapidly relaxes towards the invariant density $\widetilde{F}_0(J)$.

To complete the present paragraph, we study the resonance case for the Henon map. We assume that the unperturbed betatron tune is close to a third order resonance $3\nu_0 = 1$. For the operator $\widehat{\mathbf{L}}_1$ and for the nonlinear tune shift $\Omega'(\omega_0, J)$, we obtain

$$\widehat{\mathbf{L}}_1 = -\delta_1 \partial_a, \qquad \Omega'(\omega_0, J) = \frac{\sqrt{3}\mathcal{S}^2 J}{8}, \tag{6·11}$$

respectively. In the continuous limit equation (4·14) can be written as

$$\frac{\partial \widetilde{F}}{\partial n} = -(\omega - \Omega')\partial_a \widetilde{F} + \widehat{\mathbf{L}}_R \widetilde{F} + \frac{1}{2}\left(\widehat{\mathbf{L}}_1^2 + 2\widehat{\mathbf{L}}_1 \widehat{\mathbf{L}}_R + \widehat{\mathbf{L}}_R^2\right)\widetilde{F}, \tag{6·12}$$

where

$$\widehat{\mathbf{L}}_R = -\frac{\mathcal{S}\sqrt{2J}}{4}[2J \sin(3a)\partial_J + \cos(3a)\partial_a], \tag{6·13}$$

the frequency ω is represented by expression (4·1) up to $O(\epsilon^2)$, and the renormalized distribution function $\widetilde{F} = \widetilde{F}(a, J; n)$ is a function of the phase-space variables and the "time" n.

Equation (6·12) is a Fokker-Planck equation describing the slow evolution of the phase-space density in the case where the rotation frequency of the Henon map is close to a third order resonance.

§7. Concluding Remarks

We have applied the Renormalization Group (RG) method to study the stochastic properties of the Frobenius-Perron operator for a variety of symplectic maps. After a brief introduction and derivation of the Frobenius-Perron operator for a generic symplectic map with rotation, the case, where the unperturbed rotation frequency of the map is far from structural resonances driven by the kick perturbation has been analyzed in detail. It has been shown that up to second order in the strength of the perturbation kick, the renormalized propagator for maps with nonlinear stabilization $\left(\widehat{\mathbf{L}}_0 \neq 0\right)$ describes random wandering of the angle variable. Further, the resonance structure of a symplectic map has been investigated. It has been shown that in the case, where the unperturbed rotation frequency is close to a resonance, the reduced RG map of the Frobenius-Perron operator (or reduced phase-space density propagator) is equivalent to a discrete Fokker-Planck equation for the renormalized distribution function.

The RG method has been also applied to study the stochastic properties of the standard Chirikov-Taylor map. A nontrivial discrete analogue of the Fokker-Planck equation with a

Fokker-Planck operator acting on both canonical variables has been obtained. The latter reduces to the well-known diffusion equation in quasi-linear approximation for the angle-independent part of the distribution function.

It is worthwhile to mention that the procedure developed in the present paper (see Paragraph 4) can be applied with a slight generalization to study modulational effects in symplectic maps.

Appendix

Writing equation (3·13) in the form

$$f_{n+1}^{(1)}(a+\omega,J) - f_n^{(1)}(a,J) = \left(\widehat{\mathbf{L}}_0 + \widehat{\mathbf{L}}\right)F(a-n\omega,J), \tag{A·1}$$

we notice that the right-hand-side will give rise to two kinds of terms. Let us first consider the equation

$$\varphi_{n+1}(a+\omega,J) - \varphi_n(a,J) = \widehat{\mathbf{L}}_0 F(a-n\omega,J). \tag{A·2}$$

Taking into account the commutativity between $\widehat{\mathbf{L}}_0$ and ∂_a, we can rewrite equation (A·2) as

$$e^{\omega\partial_a}\varphi_{n+1}(a,J) - \varphi_n(a,J) = e^{-n\omega\partial_a}\widehat{\mathbf{L}}_0 F(a,J). \tag{A·3}$$

It is straightforward to verify that the solution of the last equation is

$$\varphi_n(a,J) = ne^{-n\omega\partial_a}\widehat{\mathbf{L}}_0 F(a,J) = n\widehat{\mathbf{L}}_0 F(a-n\omega,J). \tag{A·4}$$

Since the potential $V(a,J)$, the arbitrary function $F(a,J)$ and the first-order distribution function $f_n^{(1)}(a,J)$ are periodic in the angle variable a, we can represent them as a Fourier series in a

$$V(a,J) = \sum_{m\neq 0} V_m(J)e^{ima}, \qquad F(a,J) = \sum_s F_s(J)e^{isa}, \tag{A·5}$$

$$f_n^{(1)}(a,J) = \sum_k G_k^{(n)}(J)e^{ika}. \tag{A·6}$$

We substitute the above expansions into both sides of the remainder equation

$$\psi_{n+1}(a+\omega,J) - \psi_n(a,J) = \widehat{\mathbf{L}}F(a-n\omega,J), \tag{A·7}$$

and after equating similar harmonics, we obtain

$$G_k^{(n+1)}e^{ik\omega} - G_k^{(n)} = \sum_m \left[imV_m F'_{k-m} - i(k-m)V'_m F_{k-m}\right]e^{-i(k-m)n\omega}, \tag{A·8}$$

where the primes indicate differentiation with respect to the action variable J. It is straightforward to verify that the solution of equation (A·8) has the form

$$G_k^{(n)} = \sum_m \left[im \frac{V_m e^{-im\omega/2}}{2i \sin(m\omega/2)} F'_{k-m} - i(k-m) \frac{V'_m e^{-im\omega/2}}{2i \sin(m\omega/2)} F_{k-m} \right] e^{-i(k-m)n\omega}. \qquad (A·9)$$

Substituting back expression (A·9) into the expansion (A·6) for the function ψ_n and rearranging terms, we obtain

$$\psi_n(a, J) = \sum_{m,s} e^{ima} \left[im \frac{V_m e^{-im\omega/2}}{2i \sin(m\omega/2)} F'_s - is \frac{V'_m e^{-im\omega/2}}{2i \sin(m\omega/2)} F_s \right] e^{is(a-n\omega)}. \qquad (A·10)$$

Since equation (A·1) is linear, its general solution can be represented as a sum of expressions (A·4) and (A·10). To complete the derivation of the first-order solution (3·14), we note that expression (A·10) represents the Fourier expansion of the second term in (3·14), provided the potential $V_\omega(a, J)$ is given by expression (3·16).

Here, we briefly sketch the derivation of equation (3·20). The first and the last terms on the right-hand-side of equation (3·18) can be treated in a way analogous to the treatment of equation (A·2). Consider the solution of the equation

$$\Psi_{n+1}(a + \omega, J) - \Psi_n(a, J) = n\widehat{\mathbf{L}}\widehat{\mathbf{L}}_0 F(a - n\omega, J). \qquad (A·11)$$

Using the representation (A·5) and (A·6), we can write the solution of equation (A·11) as

$$\Psi_n(a, J) = \sum_k \mathcal{G}_k^{(n)}(J) e^{ika}. \qquad (A·12)$$

It can be verified by direct substitution that the functions $\mathcal{G}_k^{(n)}$ are given by the expression

$$\mathcal{G}_k^{(n)} = \sum_m (nA_{km} + B_{km}) e^{-i(k-m)n\omega}, \qquad (A·13)$$

where

$$A_{km} = im \frac{V_m e^{-im\omega/2}}{2i \sin(m\omega/2)} W'_{k-m} - i(k-m) \frac{V'_m e^{-im\omega/2}}{2i \sin(m\omega/2)} W_{k-m}, \qquad (A·14)$$

$$B_{km} = im \frac{V_m}{4 \sin^2(m\omega/2)} W'_{k-m} - i(k-m) \frac{V'_m}{4 \sin^2(m\omega/2)} W_{k-m}. \qquad (A·15)$$

Here

$$W(a, J) = \widehat{\mathbf{L}}_0 W(a, J). \qquad (A·16)$$

Thus, the desired expression for the second term on the right-hand-side of equation (3·20) is readily obtained by taking into account the Fourier expansion of the solution of equation (A·11).

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