Lagrangian and Hamiltonian Dynamics

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Lecture 1

The Passage from Newtonian to Lagrangian Dynamics

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Subjects covered

Lecture 1: The Passage from Newtonian to Lagrangian Dynamics

Lecture 2: Discussion of Lagrangian Dynamics and Passage to Hamiltonian Dynamics

Lecture 3: Discussion of Hamiltonian Dynamics and Phase Flows

Lecture 4: Canonical Transformations

Lecture 5: Hamilton-Jacobi Theory

Lecture 6: Stability and Linearisation

The most elementary form to describe the dynamics of particles is in terms of Newton's equations (Newtonian dynamics). For many applications, e.g.

- **accelerator physics**
- **celestial mechanics**
- **systems with constraints**
- **passage to quantum mechanics**
- $\cdots$

Newton's equations are not appropriate.

It is much more useful to use Lagrangian or Hamiltonian dynamics.

In the first lecture we start out from Newtonian dynamics and perform the passage to Lagrangian dynamics.

First consider *one* particle with mass m and trajectory $\vec{x}(t)$.

Assume that the force $\vec{F}(\vec{x}, \vec{v}, t)$ onto the particle is known. Then Newton's equation

$$m\,\ddot{\vec{x}}(t) = \vec{F}\big(\vec{x}(t), \dot{\vec{x}}(t), t\big)$$

gives a second-order differential equation for the trajectory.

Here and in the following, $(\ )^{\cdot} = \frac{d}{dt}$.

To every initial conditions

$$\vec{x}(0) = \vec{x}_0,\ \dot{\vec{x}}(0) = \vec{v}_0$$

there is a unique solution $\vec{x}(t)$.

This follows from the existence and uniqueness theorem for ordinary differential equations.

In Cartesian coordinates:

$$\vec{x}(t) \;=\; \sum_{i=1}^{3} x^i(t)\,\vec{e}_i\;, \qquad \dot{\vec{x}}(t) \;=\; \sum_{i=1}^{3} \dot{x}^i(t)\,\vec{e}_i\;,$$

$$\ddot{\vec{x}}(t) \;=\; \sum_{i=1}^{3} \ddot{x}^i(t)\,\vec{e}_i\;, \qquad \vec{F}(\vec{x},\vec{v},t) \;=\; \sum_{i=1}^{3} F^i(\vec{x},\vec{v},t)\,\vec{e}_i\;.$$

Newton's equations of motion read

$$m\,\ddot{x}^i(t) \;=\; F^i\big(\,\vec{x}(t),\dot{\vec{x}}(t),t\,\big)\;, \qquad i=1,2,3\,.$$

These equations are *not* covariant, i.e, they do not preserve their form if we change to curvilinear coordinates. (For proof see next page.) Curvilinear coordinates are useful because they can be adapted

- to the symmetry of the situation (in accelerator physics, e.g., choose desired path of particle as coordinate line);
- to constraints (not very relevant to accelerator physics).

Transformation from Cartesian coordinates (x^1, x^2, x^3) to new coordinates (x'^1, x'^2, x'^3):

$$x^i = x^i(x'^1, x'^2, x'^3) \; : \qquad \dot{x}^i(t) = \sum_{j=1}^{3} \frac{\partial x^i}{\partial x'^j}\big(x'^1(t), x'^2(t), x'^3(t)\big)\, \dot{x}'^j(t) \, .$$

$$\dot{\vec{x}}(t) = \sum_{i=1}^{3} \dot{x}^i(t)\, \vec{e}_i = \sum_{j=1}^{3} \dot{x}'^j(t) \underbrace{\sum_{i=1}^{3} \frac{\partial x^i}{\partial x'^j}\big(x'^1(t), x'^2(t), x'^3(t)\big)\, \vec{e}_i}_{\vec{e}\,'_j\left(x'^1(t), x'^2(t), x'^3(t)\right)} \, .$$

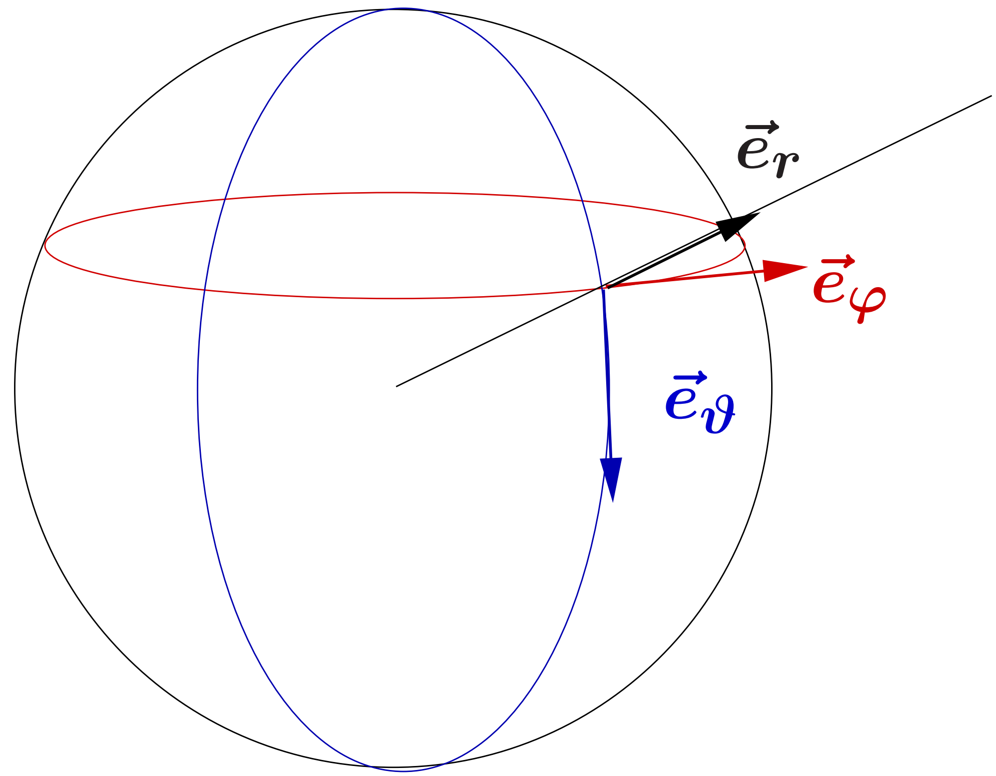
The $\vec{e}\,'_j$ are constant only if the new coordinates are rectilinear.

$$\ddot{\vec{x}}(t) = \sum_{i=1}^{3} \ddot{x}^i(t)\, \vec{e}_i = \sum_{j=1}^{3} \ddot{x}'^j(t)\, \vec{e}\,'_j\big(x'^1(t), x'^2(t), x'^3(t)\big)$$

$$+ \sum_{j=1}^{3} \sum_{i=1}^{3} \sum_{k=1}^{3} \frac{\partial^2 x^i}{\partial x'^j x'^k}\big(x'^1(t), x'^2(t), x'^3(t)\big)\, \dot{x}'^j(t)\, \dot{x}'^k(t)\, \vec{e}_i \, .$$

Thus, $\ddot{x}^i = 0 \;\Rightarrow\; \ddot{x}'^i = 0$ only if the x'^i are rectilinear coordinates.

Example: Newton's force-free equation in spherical polar coordinates $(x'^1, x'^2, x'^3) = (r, \vartheta, \varphi)$,



$$x^1 = r \sin\vartheta \cos\varphi \,,$$
$$x^2 = r \sin\vartheta \sin\varphi \,,$$
$$x^3 = r \cos\vartheta \,.$$

Then $\ddot{x}^i = 0$ is equivalent to

$$\ddot{r} - r \sin^2\!\vartheta \, \dot{\varphi}^2 - r \, \dot{\vartheta}^2 = 0 \,,$$
$$\ddot{\vartheta} + \frac{2}{r} \dot{r} \, \dot{\vartheta} - \sin\vartheta \cos\vartheta \, \dot{\varphi}^2 = 0 \,,$$
$$\ddot{\varphi} + \frac{2}{r} \dot{r} \, \dot{\varphi} + 2 \cot\vartheta \, \dot{\vartheta} \, \dot{\varphi}^2 = 0 \,.$$

We want to reformulate Newton's equation in a way that is covariant, i.e., we seek a formulation that takes the same form in any coordinates.

This will be the *Lagrangian formulation*. We do this first for some special cases before discussing the general framework of Lagrangian dynamics.

(a) Particle in a potential

Assume that the force is of the form

$$\vec{F}(\vec{x}) = -\nabla V(\vec{x})$$

with some scalar function $V(\vec{x})$. Then Newton's equation takes the form

$$m\,\ddot{\vec{x}}(t) = -\nabla V\big(\vec{x}(t)\big)\,.$$

(Example: Particle with charge q in an electrostatic field $\vec{E}(\vec{x}) = -\nabla\,\phi(\vec{x})$, where $V(\vec{x}) = q\,\phi(\vec{x})$.)

Kinetic energy: $T(\dot{\vec{x}}) = \frac{m}{2} |\dot{\vec{x}}|^2$

Potential energy: $V(\vec{x})$

Total energy $T + V$ is preserved along trajectory:

$$\frac{d}{dt}\Big(T(\dot{\vec{x}}(t)) + V(\vec{x}(t)) \Big) = \Big(m\,\ddot{\vec{x}}(t) + \nabla V(\vec{x}(t)) \Big) \cdot \dot{\vec{x}}(t) = 0\,.$$

Introduce *Lagrange function* $L = T - V$:

$$L(x^1, x^2, x^3, \dot{x}^1, \dot{x}^2, \dot{x}^3) = \frac{m}{2}\Big((\dot{x}^1)^2 + (\dot{x}^2)^2 + (\dot{x}^3)^2 \Big) - V(x^1, x^2, x^3)\,.$$

Then we have $\dfrac{\partial L}{\partial \dot{x}^i} = m\,\dot{x}^i\,,\qquad \dfrac{\partial L}{\partial x^i} = -\dfrac{\partial V}{\partial x^i}\,.$

Hence, Newton's equations $m\,\ddot{x}^i + \dfrac{\partial V}{\partial x^i} = 0$ are equivalent to the *Euler-Lagrange equations*

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i} - \frac{\partial L}{\partial x^i} = 0\,. \qquad \text{(EL)}$$

The Euler-Lagrange equations are covariant, i.e., they preserve their form under arbitrary coordinate transformations.

Of course, the Lagrange function looks different if expressed in other coordinates. What covariance means is the following:

Assume that L satisfies (EL). Make a coordinate transformation

$$x^i = x^i\big(x'^1, x'^2, x'^3\big) \, .$$

Define the new Lagrange function L' via

$$L'\big(x'^1, x'^2, x'^3, \dot{x}'^1, \dot{x}'^2, \dot{x}'^3\big) = L\big(x^1, x^2, x^3, \dot{x}^1, \dot{x}^2, \dot{x}^3\big) \, .$$

Then L' satisfies the primed Euler-Lagrange equations,

$$\frac{d}{dt}\frac{\partial L'}{\partial \dot{x}'^i} - \frac{\partial L'}{\partial x'^i} = 0 \, .$$

We now prove that the Euler-Lagrange equations are, indeed, covariant in this sense.

Assume that (EL) holds, $$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i} - \frac{\partial L}{\partial x^i} = 0\,.$$

Make coordinate transformation $x^i = x^i(x'^1, x'^2, x'^3)\,.$

$$\dot{x}^i = \sum_{j=1}^{3} \frac{\partial x^i}{\partial x'^j}\,\dot{x}'^j\,, \qquad \frac{\partial \dot{x}^i}{\partial x'^k} = \sum_{j=1}^{3} \frac{\partial^2 x^i}{\partial x'^j x'^k}\,\dot{x}'^j\,, \qquad \frac{\partial \dot{x}^i}{\partial \dot{x}'^k} = \frac{\partial x^i}{\partial x'^k}\,.$$

$$\frac{d}{dt}\frac{\partial L'}{\partial \dot{x}'^k} = \frac{d}{dt}\Big(\sum_{i=1}^{3} \frac{\partial L}{\partial \dot{x}^i}\frac{\partial \dot{x}^i}{\partial \dot{x}'^k}\Big) = \frac{d}{dt}\Big(\sum_{i=1}^{3} \frac{\partial L}{\partial \dot{x}^i}\frac{\partial x^i}{\partial x'^k}\Big) =$$

$$= \sum_{i=1}^{3}\Big(\frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i}\Big)\frac{\partial x^i}{\partial x'^k} + \sum_{i=1}^{3}\frac{\partial L}{\partial \dot{x}^i}\frac{d}{dt}\Big(\frac{\partial x^i}{\partial x'^k}\Big) =$$

$$= \sum_{i=1}^{3}\Big(\frac{\partial L}{\partial x^i}\Big)\frac{\partial x^i}{\partial x'^k} + \sum_{i=1}^{3}\frac{\partial L}{\partial \dot{x}^i}\sum_{j=1}^{3}\Big(\frac{\partial^2 x^i}{\partial x'^k \partial x'^j}\,\dot{x}'^j\Big) =$$

$$= \sum_{i=1}^{3}\Big(\frac{\partial L}{\partial x^i}\frac{\partial x^i}{\partial x'^k} + \frac{\partial L}{\partial \dot{x}^i}\frac{\partial \dot{x}^i}{\partial x'^k}\Big) = \frac{\partial L'}{\partial x'^k}\,.$$

Example: In spherical polar coordinates

$$x^1 \;=\; r \sin\vartheta \cos\varphi \,, \qquad x^2 \;=\; r \sin\vartheta \sin\varphi \,, \qquad x^3 \;=\; r \cos\vartheta \,,$$

the Lagrange function

$$L(\vec{x}^1, x^2, x^3, \dot{x}^1, \dot{x}^2, \dot{x}^3) \;=\; \frac{m}{2}\Big(\big(\dot{x}^1\big)^2 \;+\; \big(\dot{x}^2\big)^2 \;+\; \big(\dot{x}^3\big)^2 \Big) \;-\; V(x^1, x^2, x^3) \,.$$

takes the form

$$L'(r, \vartheta, \varphi, \dot{r}, \dot{\vartheta}, \dot{\varphi}) \;=\; \frac{m}{2}\Big(\dot{r}^2 \;+\; r^2\big(\dot{\vartheta}^2 \;+\; \sin^2\!\vartheta\, \dot{\varphi}^2 \big) \Big) \;-\; V'(r, \vartheta, \varphi) \,.$$

Remark: If a force field $\vec{F}(\vec{x})$ has non-vanishing curl, $\nabla \times \vec{F} \neq \vec{0}$, it cannot be written as $\vec{F} = -\nabla V$. It is then impossible to bring the equation of motion $m\ddot{\vec{x}} = \vec{F}$ into the form of the Euler-Lagrange equations.

(b) Charged particle in electromagnetic field (non-relativistic)

On a particle with charge q, an electromagnetic field $\vec{E}(\vec{x},t)$, $\vec{B}(\vec{x},t)$ exerts the *Lorentz force*. For non-relativistic motion (i.e., $|\dot{\vec{x}}| << c$), the Lorentz force equation reads

$$m\,\ddot{\vec{x}}(t) \;=\; q\Big(\,\vec{E}\big(\vec{x}(t),t\big) \;+\; \dot{\vec{x}}(t)\times\vec{B}\big(\vec{x}(t),t\big)\,\Big)\,.$$

$\vec{E}$ and $\vec{B}$ have to satisfy the Maxwell equations, in particular

$$\nabla\cdot\vec{B} \;=\; 0\,, \qquad\quad \nabla\times\vec{E} \;+\; \frac{\partial}{\partial t}\vec{B} \;=\; \vec{0}\,.$$

Owing to the first equation, $\vec{B}$ can be written as the curl of a vector potential $\vec{A}$,

$$\vec{B} \;=\; \nabla\times\vec{A}\,.$$

Owing to the second equation, $\vec{E}+\partial\vec{A}/\partial t$ can then be written as the gradient of a scalar potential ϕ,

$$\vec{E} \;=\; -\,\nabla\phi \;-\; \frac{\partial}{\partial t}\vec{A}\,.$$

We define the Lagrange function in this situation by

$$L = \frac{m}{2} \left| \dot{\vec{x}} \right|^2 - q\,\phi + q\,\vec{A} \cdot \dot{\vec{x}}\,.$$

Then the Euler-Lagrange equations

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}^i} - \frac{\partial L}{\partial x^i} = 0$$

take the form

$$\frac{d}{dt}\big(m\,\dot{\vec{x}} + q\,\vec{A} \big) + q \nabla \phi - q\,\nabla \big(\vec{A} \cdot \dot{\vec{x}} \big) = 0\,,$$

$$m\,\ddot{\vec{x}} = \underbrace{-\,q\,\nabla\phi - q\,\frac{\partial}{\partial t}\vec{A}}_{q\,\vec{E}} + \underbrace{q\left(\nabla(\vec{A}\cdot\dot{\vec{x}}) - (\dot{\vec{x}}\cdot\nabla)\vec{A} \right)}_{q\,\dot{\vec{x}}\times(\nabla\times\vec{A}) = q\,\dot{\vec{x}}\times\vec{B}}\,.$$

Thus, the Euler-Lagrange equations are equivalent to the Lorentz force equation.

Owing to the covariance of the Euler-Lagrange equations, L can be rewritten in arbitrary coordinates if this is desired.

In addition, the freedom of changing the potentials by *gauge transformations*

$$\vec{A} \longmapsto \vec{A} + \nabla f \,, \qquad \phi \longmapsto \phi - \frac{\partial}{\partial t} f$$

allows to change the Lagrangian according to

$$L \longmapsto L + q \frac{\partial}{\partial t} f + q \, \nabla f \cdot \dot{\vec{x}} = L + \frac{d}{dt}(q f)$$

with an arbitrary function $f(\vec{x}, t)$.

Such a change of the Lagrangian leaves the Euler-Lagrange equations unaltered.

(c) Charged particle in electromagnetic field (relativistic)

For relativistic motion, the Lorentz force equation reads

$$\frac{d}{dt}\left(\frac{m\,\dot{\vec{x}}(t)}{\sqrt{1-\frac{|\,\dot{\vec{x}}(t)\,|^2}{c^2}}}\right) = q\left(\vec{E}\big(\vec{x}(t),t\big) + \dot{\vec{x}}(t)\times\vec{B}\big(\vec{x}(t),t\big)\right) \qquad \text{(LF)}$$

where m is the particle's rest mass. The difference to the non-relativistic case is in the square-root on the left-hand side.

We define the Lagrange function

$$L = -m\,c^2\sqrt{1-\frac{|\,\dot{\vec{x}}(t)\,|^2}{c^2}} - q\,\phi + q\,\vec{A}\cdot\dot{\vec{x}}$$

which for small velocities, $\sqrt{1-\frac{|\,\dot{\vec{x}}(t)\,|^2}{c^2}} \approx 1-\frac{1}{2}\frac{|\dot{\vec{x}}(t)|^2}{c^2}$, reproduces the Lagrange function for the non-relativistic case up to an irrelevant additive constant.

With the relativistic Lagrange function, the Euler-Lagrange equations yield

$$\frac{d}{dt}\Big(\frac{m\,\dot{\vec{x}}}{\sqrt{1 - \frac{|\,\dot{\vec{x}}\,|^2}{c^2}}} + q\,\vec{A} \Big) + q\nabla\phi - q\,\nabla\big(\vec{A}\cdot\dot{\vec{x}}\big) = 0\,,$$

$$\frac{d}{dt}\Big(\frac{m\,\dot{\vec{x}}}{\sqrt{1 - \frac{|\,\dot{\vec{x}}\,|^2}{c^2}}} \Big) = -q\,\nabla\phi - q\,\frac{\partial}{\partial t}\vec{A} + q\,\Big(\nabla(\vec{A}\cdot\dot{\vec{x}}) - (\dot{\vec{x}}\cdot\nabla)\vec{A} \Big)\,.$$

The right-hand side is the same as for the non-relativistic case, so

$$\frac{d}{dt}\Big(\frac{m\,\dot{\vec{x}}}{\sqrt{1 - \frac{|\,\dot{\vec{x}}\,|^2}{c^2}}} \Big) = q\,\Big(\vec{E} + \dot{\vec{x}}\times\vec{B} \Big)\,,$$

i.e., the Euler-Lagrange equations of our Lagrange function are, indeed, equivalent to the relativistic Lorentz force equation (LF).

(d) Damped harmonic oscillator

Sometimes it is said that dissipative systems (i.e., systems with friction or other damping mechanisms) cannot be put into Lagrangian form.

This is true only as long as one wants to have a *time-independent* Lagrange function.

We consider the damped harmonic oscillator and show that it admits a (time-dependent) Lagrange function.

Newton's equation for a damped harmonic oscillator reads

$$m\,\ddot{x} = -k\,x - \lambda\,\dot{x}\ ,$$

where the constant k describes the restoring force and the constant λ describes the damping.

For the Lagrangian

$$L(x, \dot{x}, t) \;=\; e^{\frac{\lambda t}{m}} \left(\frac{m}{2}\, \dot{x}^2 \;-\; \frac{k}{2}\, x^2 \right)$$

the Euler-Lagrange equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} \;=\; 0$$

reads

$$\frac{d}{dt}\left(e^{\frac{\lambda t}{m}}\, m\, \dot{x} \right) \;+\; e^{\frac{\lambda t}{m}}\, k\, x \;=\; 0\,,$$

$$\cancel{e^{\frac{\lambda t}{m}}} \left(m\, \ddot{x} \;+\; \lambda\, \dot{\vec{x}} \;+\; k\, x \right) \;=\; 0\,,$$

so it is equivalent to Newton's equation.

This example demonstrates that sometimes it is necessary to work with an explicitly time-dependent Lagrange function even if the force is time-independent.

(e) Interacting particles

For N particles wthout constraints, the Lagrange function depends on $6N$ coordinates (and possibly on time): $3N$ position coordinates and $3N$ velocity coordinates.

For *non-interacting* particles, the total Lagrange function is of the form

$$L = \sum_{I=1}^{N} L_I$$

where I labels the particles and L_I depends on position and velocity coordinates of the Ith particle only.

In this case the $3N$ Euler-Lagrange equations for L decompose into the Euler-Lagrange equations for the L_I, i.e., the motion can be studied for each particle separately.

For *interacting* particles, the Lagrange function (if it exists) contains interaction terms, i.e., summands which depend on position and velocity coordinates of two or more particles. For pair-interaction

$$L = \sum_{I=1}^{N} \Big(L_I + \sum_{\substack{J=1 \\ J>I}}^{N} L_{IJ} \Big)$$

where L_{IJ} depends on position and velocity coordinates of the Ith and the Jth particle. Here is an example.

For two particles with (Newtonian) gravitational interaction, the Lagrange function is

$$L = \underbrace{\frac{m_1}{2} \, |\, \dot{\vec{x}}_1 \,|^2}_{L_1} + \underbrace{\frac{m_2}{2} \, |\, \dot{\vec{x}}_2 \,|^2}_{L_2} + \underbrace{\frac{G \, m_1 \, m_2}{|\, \vec{x}_2 - \vec{x}_1 \,|}}_{L_{12}} \, .$$

(Coulomb interaction for slowly moving charged particles is analogous.)

The six Euler-Lagrange equations are

$$\cancel{m_1}\left(\ddot{\vec{x}}_1 - \frac{G\, m_2\,(\vec{x}_2 - \vec{x}_1)}{|\,\vec{x}_2 - \vec{x}_1\,|^3}\right) = \vec{0}\,,$$

$$\cancel{m_2}\left(\ddot{\vec{x}}_2 - \frac{G\, m_1\,(\vec{x}_1 - \vec{x}_2)}{|\,\vec{x}_1 - \vec{x}_2\,|^3}\right) = \vec{0}\,.$$

Summary: In many relevant (but not in all) cases, the equations of motion can be written in the form of the Euler-Lagrange equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i} - \frac{\partial L}{\partial x^i} = 0\,, \quad i = 1, \dots, n\,.$$

The dynamics is then completely coded in the Lagrange function

$L(x^1 \dots x^n, \dot{x}^1, \dots, \dot{x}^n, t)$.

For N particles without constraints, $n = 3N$.

The Euler-Lagrange equations are covariant with respect to arbitrary coordinate transformations.