Cockroft Institute
November 2009

Longitudinal Dynamics
Tutorial - Correction

Exercise 1

From relativistic dynamics one has:

$$E = mc^2 \quad ; \quad p = mv = mc^2 \frac{v}{c^2} = \frac{E}{c}\beta \quad ; \quad pc = E\beta \quad ; \quad \beta = \left(1 - \frac{1}{\gamma^2}\right)^{\frac{1}{2}} \quad ; \quad p^2c^2 = E^2 - E_0^2$$

$$2EdE = 2c^2 pdp \quad ; \quad dE = vdp = \beta cdp$$

$$\frac{dE}{E} = \beta c \frac{dp}{E} = \beta^2 \frac{dp}{p}$$

$$\frac{d}{dt}(\beta\gamma) = \frac{dz}{dt}\frac{d}{dz}(\beta\gamma) = \beta^2 c \frac{d\gamma}{dz} + \beta c\gamma \frac{d\beta}{dz}$$

$$\beta c\gamma \frac{d\beta}{dz} = c\left(\gamma^2 - 1\right)^{1/2} \frac{d}{dz}\left(\frac{\sqrt{\gamma^2 - 1}}{\gamma}\right) = \frac{c}{\gamma^2}\frac{d\gamma}{dz}$$

$$\frac{d}{dt}(\beta\gamma) = \left(\beta^2 c + \frac{c}{\gamma^2}\right)\frac{d\gamma}{dz} = c\frac{d\gamma}{dz}$$

$$F_z = \frac{dp}{dt} = m_0 c \frac{d}{dt}(\gamma\beta) = m_0 c \frac{1}{\beta}\frac{d\gamma}{dt}$$

$$\frac{1}{\beta}\frac{d\gamma}{dt} = \frac{1}{\beta}\frac{d}{dt}\left(1 - \beta^2\right)^{-1/2} = \gamma^3 \frac{d\beta}{dt}$$

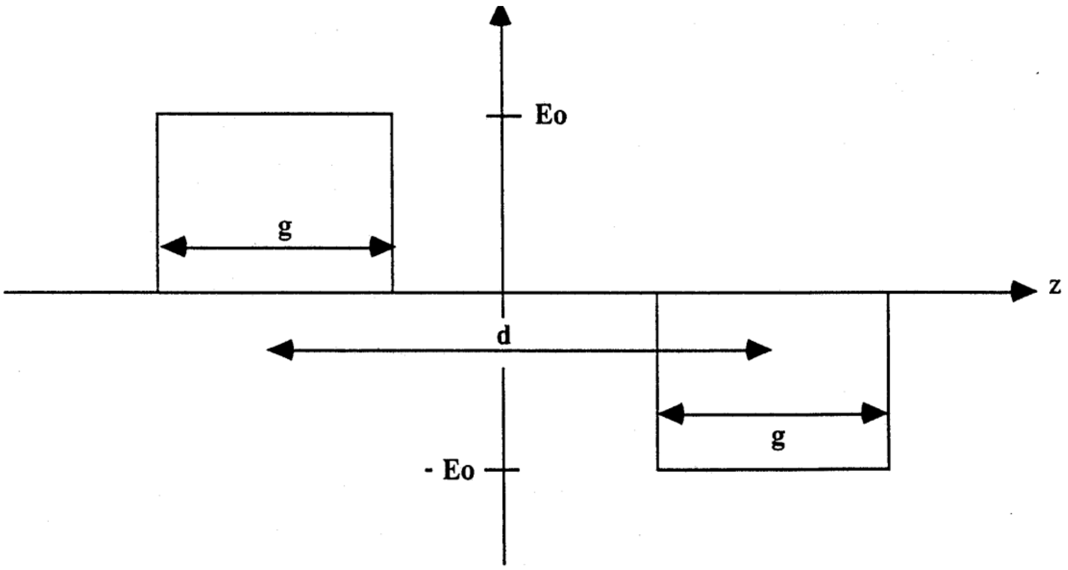
$$\frac{d\beta}{dt} = \frac{1}{m_0 c\gamma^3} F_z$$

Exercise 2

$$g = \frac{1}{2}\beta\lambda_0 = \frac{1}{2}\frac{v}{c}\frac{2\pi c}{\omega_{RF}} \quad ; \quad \theta = \frac{\omega_{RF}\, g}{v} = \pi \quad ; \quad T = \frac{\sin\frac{\vartheta}{2}}{\frac{\theta}{2}} = 0.6366$$

Exercise 3

g = 5 cm , d = 10 cm, $f_{RF} = 100\, MHz$



a) The optimum velocity is obtained when the distance between the two gaps is travelled in half an RF period:

$$v\frac{T}{2} = d \qquad v = 2*0.1*10^8 = 2*10^7\ ms^{-1}$$

b) The maximum energy gain can be written:

$$\Delta E = q\left|\int E(z)e^{j\omega\frac{z}{v}}dz\right| \qquad with \quad z<0 \quad E(z)=E_0 \quad and \quad z>0 \quad E(z)=-E_0$$

$\int = \int_{z<0} + \int_{z>0}$ and the cos terms cancel leading to:

$$\int = j\int_{z<0} E_0 \sin\frac{\omega z}{v}dz + j\int_{z>0}(-E_0)\sin\frac{\omega z}{v}dz$$

The energy gain becomes : $\Delta E = 2qE_0\left|\int_{\frac{d}{2}-\frac{g}{2}}^{\frac{d}{2}+\frac{g}{2}} \sin\left(\frac{\omega z}{v}\right)dz\right| = 2qE_0\frac{v}{\omega}\left[2\sin\left(\frac{\omega d}{2v}\right)\sin\left(\frac{\omega g}{2v}\right)\right]$

From the previous optimum: $v = \frac{2d}{T} = \frac{\omega d}{\pi} \Rightarrow \frac{\omega d}{2v} = \frac{\pi}{2}$

$$\Delta E = 4qE_0\frac{v}{\omega}\sin\left(\frac{\omega g}{2v}\right) = 2qE_0 g\frac{\sin\left(\frac{\omega g}{2v}\right)}{\frac{\omega g}{2v}}$$

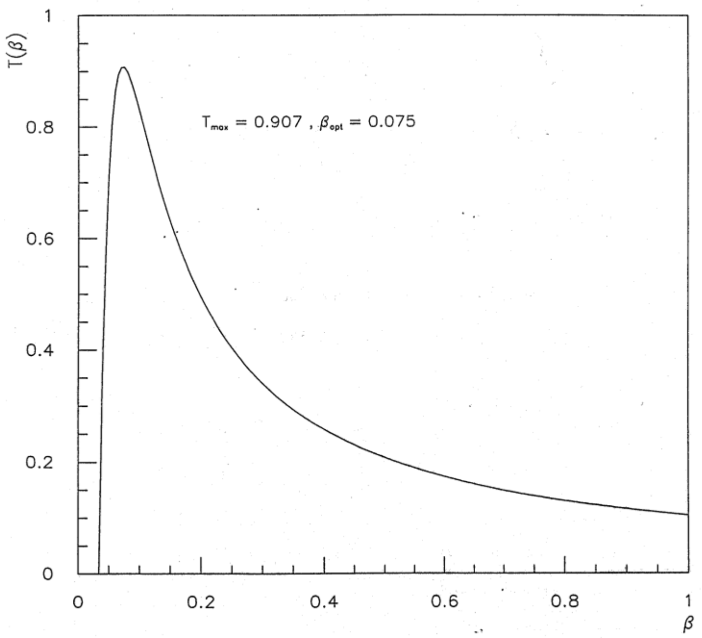
Since $E_0 = \frac{V}{g}$ and $\frac{\omega}{v} = \frac{\pi}{d}$

$$\Delta E = 2qV\frac{\sin\left(\frac{\pi g}{2d}\right)}{\left(\frac{\pi g}{2d}\right)} = 4.5\,MeV$$

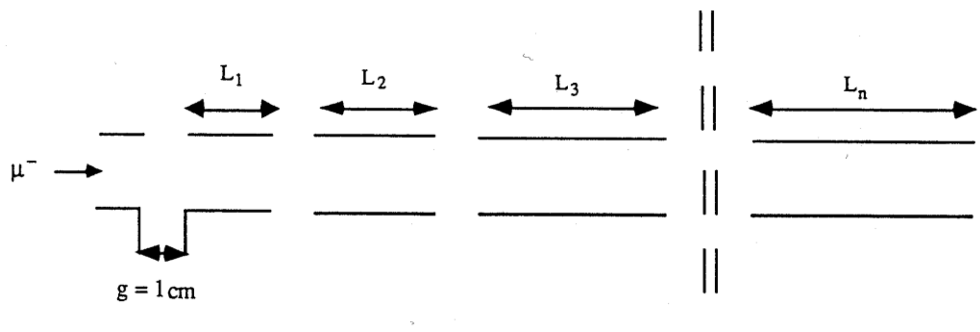
c) The transit time factor is: $T = \frac{\left|\int E(z)e^{j\omega z/v}\right|}{\int |E(z)|dz} = \frac{\Delta E}{2qE_0 g}$

$$T(v) = \frac{2v}{\omega g}\left[\sin\left(\frac{\omega d}{2v}\right)\sin\left(\frac{\omega g}{2v}\right)\right]$$

Numerical calculation has been performed has a function of $\beta = \frac{v}{c}$ and is shown on the plot below. The result justify the initial assumption.



Exercise 4



a) The kinetic energy at injection is equal to the rest energy $m_0c^2 = 105\ MeV$. Hence the total energy at the first gap entrance is :

$$E_1 = 2\ m_0c^2 = 210\ MeV$$
$$\gamma_1 = 2 = \left(1-\beta_1^2\right)^{-1/2} \quad \Rightarrow \quad \beta_1 = 0.866 \quad \Rightarrow \quad v_1 = 2.596\ 10^8\ ms^{-1}$$

The energy gain in the first gap is $\Delta W = e\hat{V}T$:

$$\hat{V} = E_0 g = 0.5\ MeV$$
$$\theta = \frac{\omega g}{v_1} = \frac{2\pi f_{RF}\, g}{v_1} = 0.121$$
$$T = \frac{\sin\theta/2}{\theta/2} = 0.9994$$
$$\Delta W = 499.7\ keV$$

b) Synchronism condition in π mode:

$$g + L_n = v_n \frac{T_{RF}}{2}$$

where v_n is the particle velocity at the entrance of L_n ;

$$L_n = c\beta_n \frac{T_{RF}}{2} - g = \frac{\lambda_0}{2}\left(1-\frac{1}{\gamma_n^2}\right)^{1/2} - g$$
$$L_n = \frac{\lambda_0}{2}\left(1-\frac{m_0^2c^4}{E_n^2}\right)^{1/2} - g$$
$$E_n = 2m_0c^2 + n\Delta W$$
$$L_n = \frac{\lambda_0}{2}\left[1-\frac{m_0^2c^4}{\left(2m_0c^2+n\Delta W\right)^2}\right]^{1/2} - g$$

At high energy $L \rightarrow L_{\max} = \frac{\lambda_0}{2} - g = 0.2998 - 0.01 = 0.2898\ m$

For the first drift $n=1 \quad \Delta W = 0.5\ MeV \quad L_1 = 0.2498\ m$

$L_{\max}$ is reached when $v = c \quad \Rightarrow \quad L_{\max} + g = \frac{\lambda_0}{2}$

c) With $\hat{V} = E_0 g = 1 MV$ synchronism is satisfied with $e\hat{V}\sin\phi_s = 499.7\, keV$:

$$\sin\phi_s = 0.4997 \quad \Rightarrow \quad \phi_s \approx 30°$$

The angular frequency of longitudinal motion (with present definition of ϕ_s) is:

$$\Omega_s^2 = \frac{eE_0\omega_{RF}\cos\phi_s}{m_0\gamma^3 v} = \left(\frac{eE_0}{m_0c^2}\right)c^2\omega_{RF}\cos\phi_s\frac{1}{\gamma^3\beta c}$$
$$\Omega_s^2 = 7.773\,10^{17}\gamma^{-3}\beta^{-1}$$
$$\Omega_s = 8.82\,10^8\gamma^{-1}\left(\gamma^2-1\right)^{-1/4}\ s^{-1}$$

In the first gap ($\gamma = 2$) one gets:

$$\Omega_s = 2\pi f_s = 335\ MHz$$

small compare to the RF frequency.

When $L \rightarrow L_{\max}$ $v \rightarrow c$ and γ gets large:

$$\Omega_s \rightarrow 8.82\,10^8\frac{1}{\gamma^{3/2}} \rightarrow 0$$

Exercise 5

Protons are accelerated in a two gaps cyclotron, which magnet has a useful diameter of 2 m with a magnetic field of 1 T:

a) The maximum kinetic energy is such that:

$p_{\max} = eB\rho_{\max}$

leading to : $(pc)_{\max} = 3.10^8\ eV$

Since $E_0 = 938\,MeV$ *and* $E^2 = E_0^2 + p^2c^2$ *one gets* $E = 984.81\,MeV$

and the maximum kinetic energy is: $W = E - E_0 = 46.8\,MeV$

b) Assuming isochronism the energy gain in each gap is:

$$\Delta E = \frac{W}{2*100} = 234\,keV$$

c) The phase variation is essentially produced at high energy when the particle is getting relativistic. Since synchronism is obtained up to the radius of 0.5 m we will consider the phase shift starts above this value. Lets introduce index "i" for the state corresponding to $\rho = 0.5\,m$ and index "f" for the final state $\rho = 1\,m$.

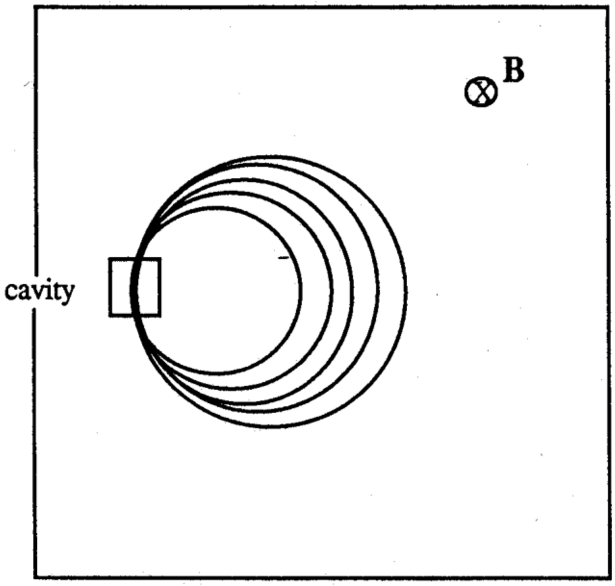
Knowing that the revolution period is : $T = \frac{2\pi m_0}{eB}\gamma$ the phase shift at high energy will be: $\Delta\phi = \phi_f - \phi_i = \omega_{RF}\left(T_f - T_i\right) = \omega_{RF}\frac{2\pi m_0}{eB}\left(\gamma_f - \gamma_i\right)$ and since synchronism at low energy means $\omega_{RF} = \frac{eB}{m_0}$ one gets the approximate phase shift:

$$\Delta\phi = 2\pi\left(\gamma_f - \gamma_i\right)$$

With $p_f c = 3.10^8\ eV$ and $p_i c = \frac{p_f c}{2}$ one gets respectively $\gamma_f = 1.0499$ and $\gamma_i = 1.0127$ leading to $\Delta\phi = 0.2337\,rd \Rightarrow 13.4\deg$, small enough to justify initial assumption.

Exercise 6

A simple microtron consists of a fixed field magnet and a single accelerating gap as shown on the figure:



a) The revolution period is:

$$\frac{2\pi m_0}{eB}\gamma$$

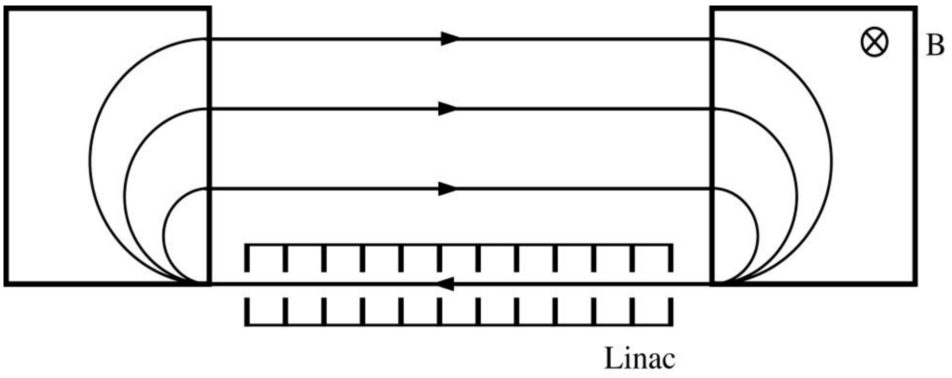
From turn to turn:

$$\frac{\Delta T}{T}=\frac{\Delta\gamma}{\gamma}$$

Lets begin with $\gamma=\gamma_0=1$ then if $\Delta\gamma$ is any integer ΔT will be a multiple of T_0 and the RF phase will rotate an integer number of 2π when the particle is back into the cavity. This is a synchronism condition (particle sees always the same phase and gets the same amount of energy gain at each turn.

b) The minimum energy gain which satisfy the concept is $\Delta\gamma=1$ or $\Delta E=E_0$. If easy to generate in a cavity a voltage of 500 kV it is rather difficult to generate a voltage of 1 GV. Hence a microtron is essentially used for accelerating electrons

c) Adding more cavity, or using an electron linac one can reach higher energies per turn. In that case the magnet is split in two pieces to leave space to the linac structure and the microtron is called “racetrack”.


B
Linac