BASIC MATHEMATICAL FORMULAE

Vectors

For vectors

$$\begin{aligned}\mathbf{u} &= (u_x, u_y, u_z) = u_x\mathbf{i} + u_y\mathbf{j} + u_z\mathbf{k}\\ \mathbf{v} &= (v_x, v_y, v_z) = v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k},\end{aligned}$$

where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are unit vectors along Cartesian axes, the *scalar* or *dot product* is

$$\mathbf{u} \cdot \mathbf{v} = u_x v_x + u_y v_y + u_z v_z \tag{1}$$

and the length of a vector is

$$|u| = \sqrt{\mathbf{u} \cdot \mathbf{u}} = \sqrt{u_x^2 + u_y^2 + u_z^2} \tag{2}$$

The *vector or cross product* is

$$\mathbf{u} \wedge \mathbf{v} = \mathbf{u} \times \mathbf{v} = (u_y v_z - u_z v_y)\mathbf{i} + (u_z v_x - u_x v_z)\mathbf{j} + (u_x v_y - u_y v_x)\mathbf{k}. \tag{3}$$

This can also be written in determinant form as

$$\mathbf{u} \wedge \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{vmatrix} \tag{4}$$

Standard identities:

$$\mathbf{u} \cdot \mathbf{v} \wedge \mathbf{w} = \mathbf{v} \cdot \mathbf{w} \wedge \mathbf{u} = \mathbf{w} \cdot \mathbf{u} \wedge \mathbf{v} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix} \tag{5}$$

$$\mathbf{u} \wedge (\mathbf{v} \wedge \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w} \tag{6}$$

Vector Calculus

The *gradient* of a scalar ϕ is

$$\operatorname{grad} \phi = \nabla \phi = \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right). \tag{7}$$

The *divergence* of a vector $\mathbf{u}$ is

$$\operatorname{div} \mathbf{u} = \nabla \cdot \mathbf{u} = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}, \tag{8}$$

and the *curl* or *rotation* of the vector is

$$\operatorname{curl} \mathbf{u} = \operatorname{rot} \mathbf{u} = \nabla \wedge \mathbf{u} = \left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z} \right)\mathbf{i} + \left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x} \right)\mathbf{j} + \left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y} \right)\mathbf{k}. \tag{9}$$

The *Laplacian* operator applied to a scalar ϕ is

$$\nabla^2 \phi = \nabla \cdot \nabla \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}, \tag{10}$$

as follows from (**??**) and (**??**).

In cylindrical coordinates (r, θ, z),

$$\begin{aligned} \nabla\phi &= \left(\frac{\partial\phi}{\partial r}, \frac{1}{r}\frac{\partial\phi}{\partial\theta}, \frac{\partial\phi}{\partial z}\right), && (11) \\ \operatorname{div}\mathbf{u} &= \nabla\cdot\mathbf{u} = \frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{\partial u_z}{\partial z}, && (12) \\ \nabla^2\phi &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial z^2}. && (13) \end{aligned}$$

Other basic results include

$$\nabla\wedge\nabla\phi = 0, \qquad \nabla\cdot\nabla\wedge\mathbf{u} = 0. \tag{14}$$

Gauss' or Divergence Theorem

For a closed surface S containing a volume V,

$$\iint_S \mathbf{a}\cdot\,\mathrm{d}\mathbf{S} = \iiint_V \nabla\cdot\mathbf{a}\,\mathrm{d}V \tag{15}$$

where $\mathrm{d}\mathbf{S}$ is in the direction of the outward pointing normal. Related results are:

$$\begin{aligned} \iiint_V \nabla\phi\,\mathrm{d}V &= \iint_S \phi\,\mathrm{d}\mathbf{S} && (16) \\ \iiint_V \nabla\wedge\mathbf{u}\,\mathrm{d}V &= -\iint_S \mathbf{u}\wedge\,\mathrm{d}\mathbf{S}. && (17) \end{aligned}$$

Stokes' Theorem

For an open surface S with an oriented boundary curve C,

$$\oint_C \mathbf{a}\cdot\,\mathrm{d}\mathbf{l} = \iint_S \nabla\wedge\mathbf{a}\cdot\,\mathrm{d}\mathbf{S} \tag{18}$$

A related result is

$$\iint_S \mathrm{d}\mathbf{S}\wedge\nabla\phi = \oint_C \phi\,\mathrm{d}\mathbf{l}. \tag{19}$$

Vector Calculus Identities

$$\begin{aligned} \nabla(\psi+\phi) &= \nabla\psi + \nabla\phi && (20) \\ \nabla(\phi\psi) &= \phi\nabla\psi + \psi\nabla\phi && (21) \\ \nabla\cdot(\mathbf{u}+\mathbf{v}) &= \nabla\cdot\mathbf{u} + \nabla\cdot\mathbf{v} && (22) \\ \nabla\cdot(\phi\mathbf{u}) &= \phi\nabla\cdot\mathbf{u} + \mathbf{u}\cdot\nabla\phi && (23) \\ \nabla\wedge(\mathbf{u}+\mathbf{v}) &= \nabla\wedge\mathbf{u} + \nabla\wedge\mathbf{v} && (24) \\ \nabla\wedge(\phi\mathbf{u}) &= \phi\nabla\wedge\mathbf{u} + \nabla\phi\wedge\mathbf{u} && (25) \\ \nabla\cdot(\mathbf{u}\wedge\mathbf{v}) &= \mathbf{v}\cdot\nabla\wedge\mathbf{u} - \mathbf{u}\cdot\nabla\wedge\mathbf{v} && (26) \\ \nabla\wedge(\nabla\wedge\mathbf{u}) &= \nabla(\nabla\cdot\mathbf{u}) - \nabla^2\mathbf{u} && (27) \\ \nabla\cdot\nabla\wedge\mathbf{u} &= 0 && (28) \\ \nabla\wedge\nabla\phi &= 0 && (29) \end{aligned}$$

Note that (**??**) *defines* ∇^2 operating on a vector. In Cartesian coordinates $\nabla^2\mathbf{u} = \left(\nabla^2 u_x, \nabla^2 u_y, \nabla^2 u_z\right)$ but in other systems, $\nabla^2\mathbf{u}$ is not in general the Laplacian operating on components separately.

Complex Numbers

With the notation $\mathrm{i} = \sqrt{-1}$, an important result is

$$\mathrm{e}^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\sin\theta. \tag{30}$$

We can write

$$\mathrm{i} = \mathrm{e}^{\mathrm{i}\pi/2} \quad \text{and} \quad \sqrt{\mathrm{i}} = \mathrm{i}^{\frac{1}{2}} = \mathrm{e}^{\mathrm{i}\pi/4} = \cos(\tfrac{1}{4}\pi) + \mathrm{i}\sin(\tfrac{1}{4}\pi) = \frac{1}{\sqrt{2}}\,(1+\mathrm{i})\,. \tag{31}$$

When dealing with time dependent quantities such as plane electromagnetic waves, it is common to use a complex form where the real part is understood. Thus, if $C = c_1 + \mathrm{i}c_2$,

$$C\mathrm{e}^{\mathrm{i}\omega t} \longleftrightarrow \mathrm{Re}\,\left(C\mathrm{e}^{\mathrm{i}\omega t}\right) = c_1\cos\omega t - c_2\sin\omega t\,.$$

Physical constants

Electron charge e	1.602177×10^{-19} coulomb
Electron mass m_e	9.108×10^{-31} kg
e/m_e	1.759×10^{11} coulomb kg^{-1}
Proton mass m_p	1.672×10^{-27} kg
m_p/m_e	1836
Vacuum permeability μ_0	$4\pi \times 10^{-7}$ henry m^{-1}
Vacuum speed of light c	2.998×10^{8} m sec^{-1}
Vacuum permittivity $\epsilon_0 = 1/(\mu_0 c^2)$	8.854×10^{-12} farad m^{-1}
Vacuum impedance $Z_0 = (\mu_0/\epsilon_0)^{\frac{1}{2}}$	376.7 ohm
Vacuum admittance $Y_0 = 1/Z_0$	2.654×10^{-3} mho
Electron rest energy $m_e c^2$	8.186×10^{-14} joule (5.110×10^{5} eV)
Classical electron radius $r_e = \mu_0 e^2/(4\pi m_e)$	2.818×10^{-15} m
Planck's constant h	6.625×10^{-34} joule sec
$\hbar = h/(2\pi)$	1.054×10^{-34} joule sec
Fine structure constant $e^2/(2\epsilon_0 hc)$	7.297×10^{-3}
Bohr radius $a_0 = 4\pi\epsilon_0\hbar^2/(m_e e^2)$	5.292×10^{-11} m
Bohr magneton $e\hbar/2m_e$	9.273×10^{-24} amp m^2
Boltzmann's constant K	1.380×10^{-23} joule deg^{-1}