

Geometry, Relativity *and Electromagnetism*

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Overview

- Euclidean 3-Space

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- Space Curves in 3 Dimensions

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- Time and Clocks

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- The Electromagnetic Tensor Field in Spacetime

The Metric in a 3-Vector Space

- A 3-frame is a linearly independent set of vectors $\{i, j, k\}$ that spans the vector space, such that any vectors x, y can be written

$$x = x_1 i + x_2 j + x_3 k$$

$$y = y_1 i + y_2 j + y_3 k$$

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- Symmetric means: $g(x, y) = g(y, x)$ for all x, y
- Non-degenerate means that if $g(x, y) = 0$ for *all* y then the vector x must be 0

The Metric in a 3-Vector Space

- Bilinear means:

$$g(a_1\mathbf{x}_1 + a_2\mathbf{x}_2, \mathbf{y}) = a_1g(\mathbf{x}_1, \mathbf{y}) + a_2g(\mathbf{x}_2, \mathbf{y})$$

for any constants a_1, a_2 and vectors $\mathbf{y}, \mathbf{x}_1, \mathbf{x}_2$

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- If the frame $\{i, j, k\}$ is *ortho-normal* in a Euclidean space then

$$g(\mathbf{x}, \mathbf{y}) = x_1y_1 + x_2y_2 + x_3y_3$$

and this is often abbreviated $\mathbf{x} \cdot \mathbf{y}$

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- Note that $L[x] = L[x']$ when $x' = R(x)$ with R representing an orthogonal transformation i.e. a rotation

Changing Ortho-normal 3-Frame


$$\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}$$

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- NB: $g(e_i, e_j) = \delta_{i,j}$ implies $g(f_i, f_j) = \delta_{i,j}$ for $i, j = 1, 2, 3$ and conversely.

Euclidean Rotations

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- It may be represented by an orthogonal matrix

Curves in Euclidean Space

- A parametrised curve in Euclidean Space is specified by prescribing a vector \mathbf{r} as a function of an arbitrary (real) parameter λ lying in some interval \mathcal{I} :

$$\mathbf{r} = \mathbf{R}(\lambda) = X(\lambda)\mathbf{i} + Y(\lambda)\mathbf{j} + Z(\lambda)\mathbf{k}$$

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- Example: Circle of radius R_0

$$X(\theta) = R_0 \cos \theta$$

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- NB. A double circle would have $0 < \theta \leq 4\pi$

Curves in Euclidean Space

- The length $L[dr]$ of the element

$d\mathbf{r} = \dot{\mathbf{R}}(\lambda) d\lambda = \{\dot{X}(\lambda)\mathbf{i} + \dot{Y}(\lambda)\mathbf{j} + \dot{Z}(\lambda)\mathbf{k}\} d\lambda$ is then

$$\begin{aligned} ds &= \sqrt{d\mathbf{r} \cdot d\mathbf{r}} \equiv \sqrt{\mathbf{g}(d\mathbf{r}, d\mathbf{r})} \\ &= \{(\dot{X}(\lambda))^2 + (\dot{Y}(\lambda))^2 + (\dot{Z}(\lambda))^2\}^{1/2} d\lambda \end{aligned}$$

Curves in Euclidean Space

- The length $L[dr]$ of the element $d\mathbf{r} = \dot{\mathbf{R}}(\lambda) d\lambda = \{\dot{X}(\lambda)\mathbf{i} + \dot{Y}(\lambda)\mathbf{j} + \dot{Z}(\lambda)\mathbf{k}\} d\lambda$ is then

$$ds = \sqrt{d\mathbf{r} \cdot d\mathbf{r}} \equiv \sqrt{\mathbf{g}(d\mathbf{r}, d\mathbf{r})}$$

$$= \{(\dot{X}(\lambda))^2 + (\dot{Y}(\lambda))^2 + (\dot{Z}(\lambda))^2\}^{1/2} d\lambda$$

- The finite length of a curve $C : \mathbf{r} = \mathbf{R}(\lambda)$, $a \leq \lambda \leq b$ is then

$$L[C] = \int_a^b ds = \int_C \sqrt{\mathbf{g}(d\mathbf{r}, d\mathbf{r})} d\lambda$$

$$= \int_a^b \{(\dot{X}(\lambda))^2 + (\dot{Y}(\lambda))^2 + (\dot{Z}(\lambda))^2\}^{1/2} d\lambda$$

Curves in Euclidean Space

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- The integral $L[C]$ is independent of the choice of parameterisation of C . It does not change if one makes the change to $\lambda' = F(\lambda)$ for a smooth monotonic F
- One may choose λ' so that $(\dot{X}(\lambda'))^2 + (\dot{Y}(\lambda'))^2 + (\dot{Z}(\lambda'))^2 = 1$. Then $ds = d\lambda'$ and

$$L[C] = \int_C ds = \int_C d\lambda'$$

with the integration over the range of λ'

Curves in Euclidean Space

- For the double circle $\dot{X} = -R_0 \sin \theta$, $\dot{Y} = R_0 \cos \theta$, $\dot{Z} = 0$
and

$$ds = \sqrt{R_0^2 \sin^2 \theta + R_0^2 \cos^2 \theta} d\theta = R_0 d\theta$$

$$L[C] = \int_0^{4\pi} R_0 d\theta = 4\pi R_0$$

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- NB The length of a space curve linking two points in space depends on the SHAPE of the curve
- The curve of SHORTEST length joining two specified points in a Euclidean space is called straight or *geodesic*

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- The Taylor Binary Pulsar

Galilean Relativity

- Physical 3-space is modelled as an Euclidean *affine* space based on a Euclidean 3-vector space in which vectors are displaced to new (arbitrary) origins by Euclidean (parallel) transport. i.e. The parallelogram law of vector addition. One simply writes $x + x_0$ to denote the affine vector to which the affine vector x is *displaced* by the 3-vector x_0

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- Newtonian dynamics identifies the motion of massive point particles in space as curves in a Euclidean Affine 3-space parameterised by a (hypothetical) universal time t . It assumes that there exist time independent directions in space (determined by distant celestial matter) that can be used to operationally define at each point in space an ortho-normal 3-frame (called a *sidereal frame* of reference). Such a frame naturally gives rise to a class of frames related to this by a time independent rotations and displacements with constant 3-velocity. Any member in this class is also called a *sidereal or inertial 3-frame*. Such frames related by translational are said to be *parallel* and determine the *natural Levi-Civita parallel transport* in space. (See I Benn and R W Tucker, An Introduction to Spinors and Geometry with Applications in Physics, (A Hilger 1987)). A particle moving with constant speed on a straight trajectory in such a frame is said to be *free*. Both sidereal frames and free particle motions are idealisations.

Galilean Relativity

- Suppose a moving point P in space is labelled by $\mathbf{r}(t)$ and $\mathbf{r}'(t')$ respectively in two inertial 3-frames moving with relative constant velocity \mathbf{v}_{obs} . Then $\mathbf{r}(t)$ and $\mathbf{r}'(t')$ are related by (the particular Galilean transformation)

$$\mathbf{r}' = \mathbf{r} - \mathbf{v}_{obs}t \quad \text{and} \quad t' = t$$

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$$\mathbf{r}' = \mathbf{r} - \mathbf{v}_{obs}t \quad \text{and} \quad t' = t$$

- Hence $\dot{\mathbf{r}}' = \dot{\mathbf{r}} - \mathbf{v}_{obs}$ and $\ddot{\mathbf{r}}' = \ddot{\mathbf{r}}$ so the Galilean acceleration of $P(t)$ is the same in both frames. Newton asserted that the laws of particle dynamics $\mathbf{F} = m\ddot{\mathbf{r}}$ are the same in all inertial frames. In which case the above Galilean transformations constitute a SYMMETRY of Newtonian dynamics.

Galilean Relativity

- Consider a source of energy at the origin of one inertial frame emitting spherical waves with speed c . Let $\mathbf{r}(t)$ be the position of a point on a spherical wave front at time t . The velocity of this point in this frame is $\dot{\mathbf{r}} = c\mathbf{n}$ where $\mathbf{n} = \frac{\mathbf{r}}{|\mathbf{r}|}$. From the above Galilean transformation the wave velocity in the second moving frame is

$$\dot{\mathbf{r}}' = c\mathbf{n} - \mathbf{v}_{obs}$$

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$$\dot{\mathbf{r}}' = c\mathbf{n} - \mathbf{v}_{obs}$$

- Hence $|\dot{\mathbf{r}}'| \neq c$ if $\mathbf{v}_{obs} \neq 0$ and the wave fronts will not appear spherical.

Maxwell's Equations

- In 1856 Maxwell predicted that, in the absence of sources, light was a system of electromagnetic vector fields satisfying

$$\nabla \cdot \mathbf{e} = 0$$

$$\nabla \cdot \mathbf{b} = 0$$

$$\nabla \times \mathbf{h} = \dot{\mathbf{d}}$$

$$\nabla \times \mathbf{e} = -\dot{\mathbf{b}}$$

where, in vacuo, $\mathbf{d} = \epsilon_0 \mathbf{e}$ and $\mathbf{b} = \mu_0 \mathbf{h}$.

Maxwell's Equations

- This system of equations admits propagating wave solutions to

$$\nabla \times \nabla \times \mathbf{e} = -\frac{1}{c^2} \ddot{\mathbf{e}}$$

$$\nabla \times \nabla \times \mathbf{b} = -\frac{1}{c^2} \ddot{\mathbf{b}}$$

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- The propagation speed c makes no reference to any particular reference 3-frame.

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- A long series of early experimental investigations indicated that the speed of light is independent of the relative motion of observer and source and the same in all directions in vacuo.
- This suggests that the Galilean symmetry of Newtonian Mechanics cannot be fundamental since it does not offer a symmetry for Maxwell's equations.

Einsteinnian Relativity

- In order to reconcile the lack of Galilean symmetry of Maxwell's equations with the apparent invariance of the speed of light for all observers, Einstein replaced the concept of Universal Newtonian time with a new unified world view of space and time. This is constructed out of the notion of SPACETIME.

Einsteinnian Relativity

- In order to reconcile the lack of Galilean symmetry of Maxwell's equations with the apparent invariance of the speed of light for all observers, Einstein replaced the concept of Universal Newtonian time with a new unified world view of space and time. This is constructed out of the notion of SPACETIME.
- In SPECIAL RELATIVITY spacetime is assumed to be a 4-dimensional real vector space on which is defined a non-degenerate, symmetric bilinear form \mathcal{G} with a Lorentzian signature.

Einsteinnian Relativity

- Denote a 4-basis in spacetime by the set of linearly independent vectors $\{\hat{t}, i, j, k\}$. The 4-vector $\sigma = ct\hat{t} + xi + yj + zk$ is said to have coordinates $\{ct, x, y, z\}$ which label an EVENT in spacetime. NB The coordinates are real. In this basis t measures time according to a clock located at $x = y = z = 0$.

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- With

$$\sigma_1 = x_0\hat{t} + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$$

$$\sigma_2 = y_0\hat{t} + y_1\mathbf{i} + y_2\mathbf{j} + y_3\mathbf{k}$$

and $\{\hat{t}, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ a \mathcal{G} -orthonormal basis the metric \mathcal{G} determines

$$\mathcal{G}(\sigma_1, \sigma_2) = -x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3$$

Einsteinnian Relativity

- The "extension" of the 4-vector $\sigma = x_0\hat{t} + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$ is

$$\mathcal{G}(\sigma, \sigma) = -(x_0)^2 + (x_1)^2 + (x_2)^2 + (x_3)^2$$

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- If $\mathcal{G}(\sigma, \sigma) < 0$ then σ defines a TIMELIKE extension
- If $\mathcal{G}(\sigma, \sigma) = 0$ then σ defines a LIGHTLIKE extension
- Any basis $\{e^0, e^1, e^2, e^3\}$ is called \mathcal{G} -orthonormal if

$$\mathcal{G}(e^a, e^b) = 1, \quad a = b = 1, 2, 3$$

$$\mathcal{G}(e^0, e^0) = -1,$$

$$\mathcal{G}(e^a, e^b) = 0, \quad a \neq b, \quad a, b = 0, 1, 2, 3$$

Lorentz Transformations

- A linear transformation Λ that relates two \mathcal{G} -orthonormal bases in spacetime is called a LORENTZ transformation.

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- If $\Lambda^a_b \mathbf{e}^b = \mathbf{f}^a$ $a, b, = 0, 1, 2, 3$ it then satisfies

$$\mathcal{G}(\mathbf{e}^a, \mathbf{e}^b) = \mathcal{G}(\mathbf{f}^a, \mathbf{f}^b) = \mathcal{G}(\Lambda^a_k \mathbf{e}^k, \Lambda^b_n \mathbf{e}^n)$$

where $\Lambda^a_k \mathbf{e}^k \equiv \Lambda^a_0 \mathbf{e}^0 + \Lambda^a_1 \mathbf{e}^1 + \Lambda^a_2 \mathbf{e}^2 + \Lambda^a_3 \mathbf{e}^3$

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- There are six elementary hyperbolic rotations corresponding to the six 2-planes (01,02,03) and (12,23,31) so the matrix Λ can be parameterized by 6 (real) parameters.

Changing Orthonormal 4-Frame

$$\begin{pmatrix} \mathbf{V} \\ e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} \cosh \chi & \sinh \chi & 0 & 0 \\ \sinh \chi & \cosh \chi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{W} \\ \mathbf{f}_1 \\ \mathbf{f}_2 \\ \mathbf{f}_3 \end{pmatrix}$$

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- NB: $g(e_a, e_b) = \delta_{a,b}$ implies $g(\mathbf{f}_a, \mathbf{f}_b) = \delta_{a,b}$ for $a, b = 0, 1, 2, 3$ and conversely.

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- If $\sigma = cte^0 + x_1e^1 + x_2e^2 + x_3e^3$ locates a spacetime event then the observer associated with the frame of reference $\{e^0, e^1, e^2, e^3\}$ interprets x_1, x_2, x_3 as the Cartesian coordinates of its spatial location and t as the local time of occurrence of this event.

Lorentz Transformations

• If t, x_1, x_2, x_3 and t', x'_1, x'_2, x'_3 are related by

$$ct' = ct \cosh \chi + x \sinh \chi$$

$$x' = ct \sinh \chi + x \cosh \chi$$

$$y' = y$$

$$z' = z$$

or

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cosh \chi & \sinh \chi & 0 & 0 \\ \sinh \chi & \cosh \chi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

for some parameter $-\infty < \chi < \infty$.

Lorentz Transformations

- Then: $-c^2(t')^2 + (x')^2 + (y')^2 + (z')^2 = -c^2(t)^2 + (x)^2 + (y)^2 + (z)^2$
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- Write $\tanh \chi = v/c$ with $|v| < c$ and use $\cosh^2 \chi - \sinh^2 \chi = 1$ and $\tanh \chi = \sinh \chi / \cosh \chi$ so

$$\cosh^2 \chi = \frac{1}{1 - \tanh^2 \chi} = \frac{1}{1 - \frac{v^2}{c^2}}$$

$$\sinh^2 \chi = \cosh^2 \chi - 1 = \frac{\frac{v^2}{c^2}}{1 - \frac{v^2}{c^2}}$$

Hence

$$t' = \frac{t + \frac{vx}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} \quad \text{and} \quad x' = \frac{x + vt}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Lorentz Transformations

• If $|\frac{v}{c}| < 1$

$$t' \simeq t$$

$$x' \simeq x + vt$$

and the Galilean transformations are recovered if we identify v with the relative Newtonian speed relating the two frames of reference.

Observers in Spacetime

- If $C(\lambda)$ is a parameterized curve in spacetime, $\dot{C} \equiv \frac{dC}{d\lambda}(\lambda)$ is called its tangent vector at the event $C(\lambda)$. The curve is said to be time-like where its tangent vector is time-like :

$$\mathcal{G}(\dot{C}, \dot{C}) < 0$$

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- The extension (or duration) of a time-like curve linking events $C(\lambda_1)$ and $C(\lambda_2)$ is

$$L[C] = \int_{\lambda_1}^{\lambda_2} |\mathcal{G}(\dot{C}, \dot{C})|^{\frac{1}{2}} d\lambda$$

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$$L[C] = \int_{\lambda_1}^{\lambda_2} d\lambda = \lambda_2 - \lambda_1$$

NB This depends on C .

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- An (ideal) observer is defined to be a proper-time parameterized future-pointing timelike curve on spacetime. The curve is the observer's worldline or history. It is a FUNDAMENTAL ASSUMPTION that observer's on arbitrary observer curves have clocks that operationally determine proper time along their worldline. Thus there is NO UNIQUE proper-time interval between two events that can be joined by a family of observer curves. Each curve will in general assign (measure) a different proper-time interval since each curve has a different extension.

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- Suppose such a rod appears to have a uniform speed v along the x axis of some observer, i.e. the history of one end of the rod is $x = vt$ in terms of the observer's clock measuring time t , while the history of the other end is $x = vt + L_m$.

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- The observer considers that at any instant t the moving rod has length L_m .

The Concept of Length in Relativity

- By drawing the world lines of the stationary observer, the two end of the rod, moving relative to the observer with speed v and the worldlines of point light pulses making a return flight between the ends of the rod with speed 1 show that

$$L_m = L_0 \sqrt{(1 - v^2)}$$

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Relativistic Particle Kinematics

- A tangent vector \mathbf{V} to a curve in spacetime is called a 4-velocity vector if $\mathcal{G}(\mathbf{V}, \mathbf{V}) = -c^2$. e.g

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$$\mathcal{E}_{obs} = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}}$$

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Photons

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- The tangent vector to the light-like history Γ_0 of any photon is then

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- Photons are observed as quantum processes by photon detectors. In terms of \mathcal{E}_{obs} the frequency of a photon detected by an idealized detector \mathbf{C} is

$$\nu_{obs} = \frac{\mathcal{E}_{obs}}{h}$$

where h is Planck's constant.

Antisymmetric Tensors in Spacetime

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Antisymmetric Tensors in Spacetime

- Let $\{e^0, e^1, e^2, e^3\}$ be a frame on spacetime
- A tensor basis for antisymmetric (degree 2) tensors in 4 dimensions is

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where

$$e^a \wedge e^b = -e^b \wedge e^a \quad a, b = 0, 1, 2, 3$$

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- NB: $e^0 \wedge e^0 = e^1 \wedge e^1 = e^2 \wedge e^2 = e^3 \wedge e^3 = 0$

Antisymmetric Tensors in Spacetime

- A general degree 2 antisymmetric tensor \mathcal{F} has 6 independent components $\{\mathcal{F}_{ab} = -\mathcal{F}_{ba}\}$ given in the above basis by

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- A *tensor field* in spacetime is an assignment of a tensor at each event in spacetime

The Maxwell Field Tensor

- The (relativistic) electromagnetic field tensor F is an antisymmetric degree 2 tensor field in spacetime that satisfies Maxwell's spacetime field equations. In an observer frame $\{V, e^1, e^2, e^3\}$ with $\mathcal{G}(V, V) = -1$ we may write

$$F = \mathcal{E}_{obs_V} \wedge V + \mathcal{B}_{obs_V}$$

where

$$\mathcal{E}_{obs_V} = \mathcal{E}_{obs_V 1} e^1 + \mathcal{E}_{obs_V 2} e^2 + \mathcal{E}_{obs_V 3} e^3$$

is the electric field at any event \mathcal{P} observed by V at \mathcal{P} and

$$\mathcal{B}_{obs_V} = \mathcal{B}_{obs_V 1} e^2 \wedge e^3 + \mathcal{B}_{obs_V 2} e^3 \wedge e^1 + \mathcal{B}_{obs_V 3} e^1 \wedge e^2$$

is the magnetic field at any event \mathcal{P} observed by V at \mathcal{P}

The Maxwell Field Tensor

- If another observer has frame $\{\mathbf{W}, \mathbf{f}^1, \mathbf{f}^2, \mathbf{f}^3\}$ with $\mathcal{G}(\mathbf{W}, \mathbf{W}) = -1$ where

$$\begin{pmatrix} \mathbf{V} \\ \mathbf{e}^1 \\ \mathbf{e}^2 \\ \mathbf{e}^3 \end{pmatrix} = \begin{pmatrix} \cosh \chi & \sinh \chi & 0 & 0 \\ \sinh \chi & \cosh \chi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{W} \\ \mathbf{f}^1 \\ \mathbf{f}^2 \\ \mathbf{f}^3 \end{pmatrix}$$

for some rapidity parameter $-\infty < \chi < \infty$ that boosts in V 's direction \mathbf{e}^1 , then, at any event \mathcal{P} , \mathbf{W} will observe the electric field

$$\mathcal{E}_{obs_W} = \mathcal{E}_{obs_W 1} \mathbf{e}^1 + \mathcal{E}_{obs_W 2} \mathbf{e}^2 + \mathcal{E}_{obs_W 3} \mathbf{e}^3$$

and magnetic field

$$\mathcal{B}_{obs_W} = \mathcal{B}_{obs_W 1} \mathbf{e}^2 \wedge \mathbf{e}^3 + \mathcal{B}_{obs_W 2} \mathbf{e}^3 \wedge \mathbf{e}^1 + \mathcal{B}_{obs_W 3} \mathbf{e}^1 \wedge \mathbf{e}^2$$

where

$$F = \mathcal{E}_{obs_V} \wedge \mathbf{V} + \mathcal{B}_{obs_V} = \mathcal{E}_{obs_W} \wedge \mathbf{W} + \mathcal{B}_{obs_W}$$

Example

- Suppose V observes F to be a pure magnetic field \mathcal{B}_{obs_V} at \mathcal{P} orthogonal to his or her spatial direction e^1 :

$$\mathcal{B}_{obs_V} = \mathcal{B}_{obs_V 2} e^3 \wedge e^1 + \mathcal{B}_{obs_V 3} e^1 \wedge e^2$$

Then in this frame at event \mathcal{P}

$$F = \mathcal{B}_{obs_V 2} e^3 \wedge e^1 + \mathcal{B}_{obs_V 3} e^1 \wedge e^2$$

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- But

$$e^1 = \sinh \chi \mathbf{W} + \cosh \chi \mathbf{f}^1$$

$$e^2 = \mathbf{f}^2 \quad \text{and} \quad e^3 = \mathbf{f}^3$$

so in terms of the frame $\{\mathbf{W}, \mathbf{f}^1, \mathbf{f}^2, \mathbf{f}^3\}$

$$\begin{aligned} F &= \mathcal{B}_{obs_V 2} \mathbf{f}^3 \wedge (\sinh \chi \mathbf{W} + \cosh \chi \mathbf{f}^1) + \mathcal{B}_{obs_V 3} (\sinh \chi \mathbf{W} + \cosh \chi \mathbf{f}^1) \mathbf{f}^2 \\ &= (\mathcal{B}_{obs_V 2} \sinh \chi \mathbf{f}^3 - \mathcal{B}_{obs_V 3} \sinh \chi \mathbf{f}^2) \wedge \mathbf{W} + \mathcal{B}_{obs_V 3} \cosh \chi \mathbf{f}^1 \wedge \mathbf{f}^2 \\ &\quad + \mathcal{B}_{obs_V 2} \cosh \chi \mathbf{f}^3 \wedge \mathbf{f}^1 = \mathcal{E}_{obs_W} \wedge \mathbf{W} + \mathcal{B}_{obs_W} \end{aligned}$$

Example

● So

$$\mathcal{E}_{obs_W} = \mathcal{B}_{obs_{V_2}} \sinh \chi \mathbf{f}^3 - \mathcal{B}_{obs_{V_3}} \sinh \chi \mathbf{f}^2$$

$$\mathcal{B}_{obs_W} = \mathcal{B}_{obs_{V_3}} \cosh \chi \mathbf{f}^1 \wedge \mathbf{f}^2 + \mathcal{B}_{obs_{V_2}} \cosh \chi \mathbf{f}^3 \wedge \mathbf{f}^1$$

$$\mathcal{E}_{obs_{W_1}} = 0, \quad \mathcal{E}_{obs_{W_2}} = -\mathcal{B}_{obs_{V_3}} \sinh \chi, \quad \mathcal{E}_{obs_{W_3}} = \mathcal{B}_{obs_{V_2}} \sinh \chi$$

$$\mathcal{B}_{obs_{W_1}} = 0, \quad \mathcal{B}_{obs_{W_2}} = \mathcal{B}_{obs_{V_2}} \cosh \chi, \quad \mathcal{B}_{obs_{W_3}} = \mathcal{B}_{obs_{V_3}} \cosh \chi$$

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$$\mathcal{E}_{obs_W} = \mathcal{B}_{obs_V 2} \sinh \chi \mathbf{f}^3 - \mathcal{B}_{obs_V 3} \sinh \chi \mathbf{f}^2$$

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• If $|\frac{v}{c}| \ll 1$ then $\sinh \chi \simeq \chi \simeq \frac{v}{c}$ and $\cosh \chi \simeq 1$ so to first order

$$\mathcal{B}_{obs_W} \simeq \mathcal{B}_{obs_V}$$

$$\mathcal{E}_{obs_W 2} \simeq -\frac{v}{c} \mathcal{B}_{obs_V 3} \quad \text{and} \quad \mathcal{E}_{obs_W 3} \simeq \frac{v}{c} \mathcal{B}_{obs_V 2}$$

i.e.

$$\mathcal{E}_{obs_W} \simeq \frac{\mathbf{v}_{obs}}{c} \times \mathcal{B}_{obs_V}$$

and observer W perceives an additional electrical field \mathcal{E}_{obs_W} as well as \mathcal{B}_{obs_W}