# Cockcroft Theory Lectures

## Solving the wave equation equation is cylindrical and spherical coordinate systems.

Bessel's functions and Spherical Harmonics

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# SECTION 1

## Coordinate systems and the Laplacian

- Why $\nabla$ is different in different coordinate systems.
- Plane polar coordinates: Grad, Div and the Laplacian.
- Cylindrical polar coordinates: Grad, Div, Curl and the Laplacian.
- Spherical polar coordinates: Grad, Div, Curl and the Laplacian. (Lecture 3)

# Grad, Div and Laplacian

- Recall what grad, div and Laplacian are in two dimensions.

$$\nabla\psi = \frac{\partial\psi}{\partial x}\underline{i} + \frac{\partial\psi}{\partial y}\underline{j}\,, \quad \nabla\cdot(v_1\underline{i} + v_2\underline{j}) = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} \quad \text{and} \quad \triangle\psi = \nabla\cdot\nabla\psi$$

- Grad, div and Laplacian are all linear with respect to addition and multiplication by a scalar constant $\lambda$. (Show)

$$\nabla(\psi_1 + \psi_2) = \nabla\psi_1 + \nabla\psi_2\,, \quad \nabla\cdot(\underline{v}_1 + \underline{v}_2) = \nabla\cdot\underline{v}_1 + \nabla\cdot\underline{v}_2\,,$$
$$\triangle(\psi_1 + \psi_2) = \triangle\psi_1 + \triangle\psi_2$$
$$\nabla(\lambda\psi) = \lambda\nabla\psi\,, \quad \nabla\cdot(\lambda\underline{v}) = \lambda\nabla\cdot\underline{v} \quad \text{and} \quad \triangle(\lambda\psi) = \lambda\triangle\psi$$

- However the effect of multiplication by a scalar field, then we must use the appropriate product rule. (Show)

$$\nabla(f\psi) = f\nabla\psi + \psi\nabla f \quad \text{and} \quad \nabla\cdot(f\underline{v}) = f\nabla\cdot\underline{v} + (\nabla f)\cdot\underline{v}$$

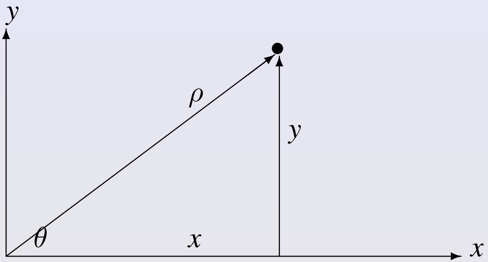
- We also have the function of a function rule. For example if $\psi$ and $\underline{v}$ are functions of two scalar fields $f$ and $g$ then (Show)

$$\nabla(\psi(f,g)) = \frac{\partial\psi}{\partial f}\nabla f + \frac{\partial\psi}{\partial g}\nabla g \quad \text{and} \quad \nabla\cdot(\underline{v}(f,g)) = \frac{\partial\underline{v}}{\partial f}\cdot\nabla f + \frac{\partial\underline{v}}{\partial g}\cdot\nabla g$$

# Plane polar coordinates

- Plane polar coordinates are given by the pair $(\rho, \theta)$ where $\rho \in \mathbb{R}$, $\rho \geq 0$ and $\theta \in \mathbb{R}$, $0 \leq \theta < 2\pi$.
- The relationship between $(\rho, \theta)$ and Cartesian coordinates is given by

$$x = \rho \cos\theta \qquad \text{and} \qquad y = \rho \sin\theta$$




- These can be inverted

$$\rho = \sqrt{x^2 + y^2} \qquad \text{and} \qquad \theta = \arctan(y/x)$$

where we need to make sure $\theta$ is in the correct quadrant.

# Scalar and vector fields

- Since we have two coordinate systems, we can represent a scalar field in two different ways.
- Let $\psi = \psi(x, y)$ be a real scalar field. There is an associated scalar field $\hat{\psi} = \hat{\psi}(\rho, \theta)$, which is defined by

$$\hat{\psi}(\rho, \theta) = \psi(\rho \cos\theta, \rho \sin\theta) = \psi(x, y)$$

- Using the chain rule we have

$$\frac{\partial \hat{\psi}}{\partial \rho} = \frac{\partial x}{\partial \rho}\frac{\partial \psi}{\partial x} + \frac{\partial y}{\partial \rho}\frac{\partial \psi}{\partial y} = \cos\theta \frac{\partial \psi}{\partial x} + \sin\theta \frac{\partial \psi}{\partial y}$$

  and

$$\frac{\partial \hat{\psi}}{\partial \theta} = \frac{\partial \psi}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial \psi}{\partial y}\frac{\partial y}{\partial \theta} = \rho \sin\theta \frac{\partial \psi}{\partial x} - \rho \cos\theta \frac{\partial \psi}{\partial y}$$

- Since the coordinates $(x, y)$ and $(\rho, \theta)$ "represent the same point" but in different coordinate systems, people tend to write $\psi = \hat{\psi}$.
- Likewise for vector fields we write $\underline{v}(\rho, \theta) = \underline{v}(x, y)$.

# Plane polar coordinates: Orthonormal basis

- Given a scalar field $\psi = \psi(\rho, \theta)$ then

$$\nabla\psi = \frac{\partial\psi}{\partial\rho}\nabla\rho + \frac{\partial\psi}{\partial\theta}\nabla\theta$$

- There are two important vector fields $\underline{e}_\rho = \underline{e}_\rho(\rho, \theta)$ and $\underline{e}_\theta = \underline{e}_\theta(\rho, \theta)$ defined by

$$\underline{e}_\rho = \frac{\nabla\rho}{\|\nabla\rho\|} \qquad \text{and} \qquad \underline{e}_\theta = \frac{\nabla\theta}{\|\nabla\theta\|}$$

- We can show that (Show)

$$\underline{e}_\rho = \cos\theta\,\underline{i} + \sin\theta\,\underline{j} \qquad \text{and} \qquad \underline{e}_\theta = -\sin\theta\,\underline{i} + \cos\theta\,\underline{j}$$

- Furthermore (Show)

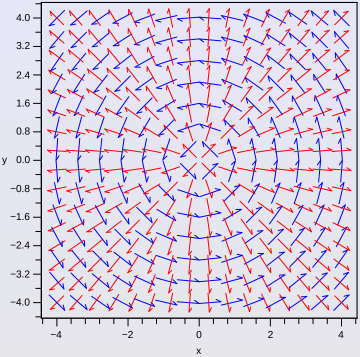
$$\nabla\rho = \underline{e}_\rho \qquad \text{and} \qquad \nabla\theta = \frac{1}{\rho}\underline{e}_\theta$$

- Hence

$$\nabla\psi = \frac{\partial\psi}{\partial\rho}\underline{e}_\rho + \frac{\partial\psi}{\partial\theta}\frac{1}{\rho}\underline{e}_\theta$$

# Plane polar coordinates: Orthonormal basis

- $$\underline{e}_\rho = \cos\theta \underline{i} + \sin\theta \underline{j} \qquad \text{and} \qquad \underline{e}_\theta = -\sin\theta \underline{i} + \cos\theta \underline{j}$$




$\underline{e}_\rho$ is red
$\underline{e}_\theta$ is blue

- Observe that $\underline{e}_\rho$ and $\underline{e}_\theta$ depend on the position.
- Observe that $\underline{e}_\rho$ and $\underline{e}_\theta$ are orthonormal, that is $\underline{e}_\rho \cdot \underline{e}_\theta = 0$ and $\underline{e}_\rho \cdot \underline{e}_\rho = \underline{e}_\theta \cdot \underline{e}_\theta = 1$. Thus we have an orthonormal vector field basis.

# Plane polar coordinates: Div

- From the function of a function rule the divergence of a vector field $\underline{v}(\rho,\theta) = v_\rho \underline{e}_\rho + v_\theta \underline{e}_\theta$ is given by (Why?)

$$\nabla \cdot \underline{v} = v_\rho \nabla \cdot \underline{e}_\rho + \nabla v_\rho \cdot \underline{e}_\rho + v_\theta \nabla \cdot \underline{e}_\theta + \nabla v_\theta \cdot \underline{e}_\theta$$

- Now

$$\nabla v_\rho = \frac{\partial v_\rho}{\partial \rho}\underline{e}_\rho + \frac{1}{\rho}\frac{\partial v_\rho}{\partial \theta}\underline{e}_\theta \quad \text{and} \quad \nabla v_\theta = \frac{\partial v_\theta}{\partial \rho}\underline{e}_\rho + \frac{1}{\rho}\frac{\partial v_\theta}{\partial \theta}\underline{e}_\theta$$

- and (Show)

$$\nabla \cdot \underline{e}_\rho = \frac{1}{\rho} \qquad \text{and} \qquad \nabla \cdot \underline{e}_\theta = 0$$

hence

$$\nabla \cdot \underline{v} = \frac{v_\rho}{\rho} + \frac{\partial v_\rho}{\partial \rho} + \frac{1}{\rho}\frac{\partial v_\theta}{\partial \theta}$$

$$\nabla \cdot \underline{v} = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\, v_\rho\right) + \frac{1}{\rho}\frac{\partial v_\theta}{\partial \theta}$$

# Plane polar coordinates: Laplacian

- $$\nabla\psi = \frac{\partial\psi}{\partial\rho}\underline{e}_\rho + \frac{1}{\rho}\frac{\partial\psi}{\partial\theta}\underline{e}_\theta$$

  and

  $$\nabla\cdot\underline{v} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho\, v_\rho) + \frac{1}{\rho}\frac{\partial v_\theta}{\partial\theta}$$

- Hence (Show)

  $$\triangle\psi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\,\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\theta^2}$$

# Cylindrical Coordinates: Grad, Div and Laplacian

- Cylindrical Coordinates are given by $(\rho, \theta, z)$ where

$$\rho \geq 0\,, \qquad 0 \leq \theta < 2\pi \qquad \text{and} \qquad -\infty < z < \infty$$

- The relationship with Cartesian coordinates $(x, y, z)$ is given by

$$x = \rho\cos\theta\,, \qquad y = \rho\sin\theta \qquad \text{and} \qquad z = z$$

  The orthogonal vector basis is given by

$$\underline{e}_\rho = \frac{\nabla\rho}{\|\nabla\rho\|}\,, \qquad \underline{e}_\theta = \frac{\nabla\theta}{\|\nabla\theta\|} \qquad \text{and} \qquad \underline{e}_z = \frac{\nabla z}{\|\nabla z\|}$$

- Giving

$$\underline{e}_\rho = \cos\theta\underline{i} + \sin\theta\underline{j}\,, \qquad \underline{e}_\theta = -\sin\theta\underline{i} + \cos\theta\underline{j} \qquad \text{and} \qquad \underline{e}_z = \underline{k}$$

# Cylindrical Coordinates: Grad, Div and Laplacian

- Let $\psi = \psi(\rho, \theta, z)$ and $\underline{v} = v_\rho \underline{e}_\rho + v_\theta \underline{e}_\theta + v_z \underline{e}_z$, where $v_\rho = v_\rho(\rho, \theta, z)$ etc.
- The Grad, Div and Laplacian are given by

$$\nabla \psi = \frac{\partial \psi}{\partial \rho}\underline{e}_\rho + \frac{1}{\rho}\frac{\partial \psi}{\partial \theta}\underline{e}_\theta + \frac{\partial \psi}{\partial z}\underline{e}_z$$

$$\nabla \cdot \underline{v} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho\, v_\rho) + \frac{1}{\rho}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z}$$

$$\triangle \psi = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\, \frac{\partial \psi}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 \psi}{\partial \theta^2} + \frac{\partial^2 \psi}{\partial z^2}$$

- The curl is given by

$$\nabla \times \underline{v} = \left(\frac{1}{\rho}\frac{\partial v_z}{\partial \theta} - \frac{\partial v_\theta}{\partial z}\right)\underline{e}_\rho + \left(\frac{\partial v_\rho}{\partial z} - \frac{\partial v_z}{\partial \rho}\right)\underline{e}_\theta + \frac{1}{\rho}\left(\frac{\partial\left(\rho v_\theta\right)}{\partial \rho} - \frac{\partial v_\rho}{\partial \theta}\right)\underline{e}_z$$

# SECTION 2

## Wave Equation on the disc and Bessel's functions

- The disc wave equation.
- Separation of variables and the time equation.
- Disc harmonics and Bessel's functions.
- Eigen properties of disc harmonics.
- Orthogonality properties of disc harmonics.
- Scalar waves in cylindrical wave guides.

# Wave on the disc

- The wave equation on the disc for the scalar $\psi = \psi(t, \rho, \theta)$ is given by

$$\frac{\partial^2 \psi}{\partial t^2} - \triangle_{\text{disc}} \psi = 0$$

where the Laplace operator in plane polar coordinates is

$$\triangle_{\text{disc}} \psi = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial \psi}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 \psi}{\partial \theta^2}$$

- We use separation of variables $\psi(t, \rho, \theta) = T(t) \mathcal{J}(\rho, \theta)$ to give (show)

$$\frac{d^2 T}{dt^2} + \omega^2 T = 0 \qquad \text{and} \qquad \triangle_{\text{disc}} \mathcal{J}(\rho, \theta) + \omega^2 \mathcal{J} = 0$$

- Clearly we can solve the $T$ equation by setting

$$T(t) = e^{i\omega t} \qquad \text{and} \qquad e^{-i\omega t}$$

where $\omega > 0$.

# Separation of the disc Laplace operator.

- We are left with looking at the eigenvalues and eigenfunctions of the disc Laplace operator

$$\triangle_{\text{disc}}\mathcal{J} = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\mathcal{J}}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\mathcal{J}}{\partial\theta^2} = -\omega^2\mathcal{J}$$

  where $\mathcal{J} = \mathcal{J}(\rho,\theta)$

- This we solve by separation of variables, Setting

$$\mathcal{J}(\rho,\theta) = \mathtt{P}(\rho)\Theta(\theta)$$

- Using $-m^2$ as the constant of separation gives (Show)

$$\frac{d^2\Theta}{d\theta^2} + m^2\Theta = 0$$

$$\rho\frac{d}{d\rho}\left(\rho\frac{d\mathtt{P}}{d\rho}\right) + (\omega^2\rho^2 - m^2)\mathtt{P} = 0$$

# Separation of the disc Laplace operator, The $\Theta$ equation.

- $$\frac{d^2\Theta}{d\theta^2} + m^2\Theta = 0$$
- The $\Theta$ equation is solved by setting
$$\Theta(\theta) = \Theta_m(\theta) = A_m e^{im\theta}$$
- Since $\theta$ is an angle so that $\Theta(\theta + 2\pi) = \Theta(\theta)$ thus $m \in \mathbb{Z}$.
- For real solution it is necessary to set
$$\Theta_m = \begin{cases} A_m \cos(m\theta) + B_m \sin(m\theta) & \text{if } m > 0 \\ A_0 & \text{if } m = 0 \end{cases}$$

# Separation of the disc Laplace operator, The $\mathtt{P}$ equation.

- $$\rho \frac{d}{d\rho}\left(\rho \frac{d\mathtt{P}}{d\rho}\right) + (\omega^2\rho^2 - m^2)\mathtt{P} = 0$$
- We transform the $\mathtt{P}$ equation into the Bessel's equation by setting
$$\rho = \frac{x}{\omega} \qquad \text{and} \qquad \mathtt{P}(\rho) = y(x)$$
- Thus
$$\frac{d}{d\rho} = \omega \frac{d}{dx}$$
Giving
$$x \frac{d}{dx}\left(x \frac{dy}{dx}\right) + (x^2 - m^2)y = 0$$
where $y = y(x)$ and $m \in \mathbb{Z}$.
- This is **Bessel's equation** of order $m$.

# Bessel functions

- We have obtained Bessel's equation, $y = y(x)$ and $m \in \mathbb{Z}$.

$$x\frac{d}{dx}\left(x\frac{dy}{dx}\right) + (x^2 - m^2)y = 0$$

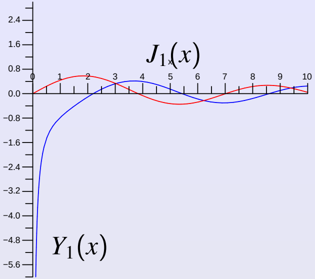
- These are solved by Bessel functions.
- Bessel's equation is a second order ordinary differential equation. Therefore we know the answer consists of two independent solutions.

$$y(x) = AJ_m(x) + BY_m(x)$$

where we call $J_m(x)$ a **Bessel function of the first kind** and $Y_m(x)$ a **Bessel function of the second kind**.

# Bessel functions

- In the limit $x \longrightarrow 0$,



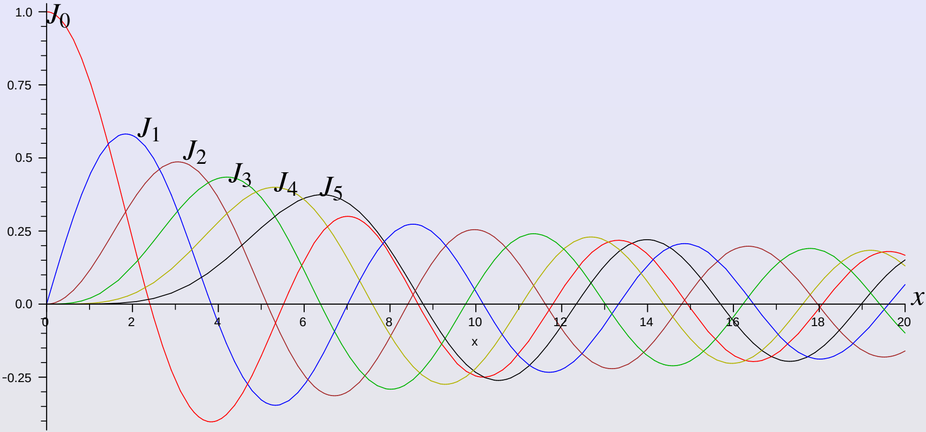

$$\lim_{x \longrightarrow 0} J_m(x) = J_m(0) \quad \text{is a finite quantity}$$

$$\lim_{x \longrightarrow 0} Y_m(x) = \infty$$

- Since $x = 0$ corresponds to $\rho = 0$ is the centre of the disc, we require that the coefficient of $Y_m(x)$ is zero.
- Thus $y(x) = AJ_m(x)$.
- After a long calculation we can show that

$$J_m(x) = \sum_{s=0}^{\infty} \frac{(-1)^s x^{m+2s}}{2^{m+2s} s!(m+s)!}$$

Plotting the first 6 Bessel functions (of the first kind) gives.




Observe that each Bessel function has an infinite set of roots (zeros).

# Recap the wave equation on the disc

- For fixed frequency $\omega$

$$\triangle_{\text{disc}}\mathcal{J} = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\mathcal{J}}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\mathcal{J}}{\partial\theta^2} = -\omega^2\mathcal{J}$$

- Since we can superimpose solutions the general solution is given by

$$\mathcal{J}(\rho,\theta) = \sum_{m=-\infty}^{\infty} A_m e^{im\theta} J_m(\omega\rho)$$

- Thus the solution to the wave equation $\frac{\partial\psi}{\partial t^2} + \triangle_{\text{disc}}\psi = 0$ is a sum of solutions of the form

$$\psi(t,\rho,\theta) = e^{i\omega t}e^{im\theta}J_m(\omega\rho) \qquad \text{and} \qquad \psi(t,\rho,\theta) = e^{-i\omega t}e^{im\theta}J_m(\omega\rho)$$

  where $m \in \mathbb{Z}$ and $\omega > 0$.

- To establish the possible values of $\omega$ we need the boundary conditions.

# Boundary conditions

- Let us consider a cylinder of radius $R$. We impose boundary conditions that $\psi = 0$ when $\rho = R$. That is

$$\psi(t, R, \theta) = 0 \qquad \text{for all } t, \theta$$

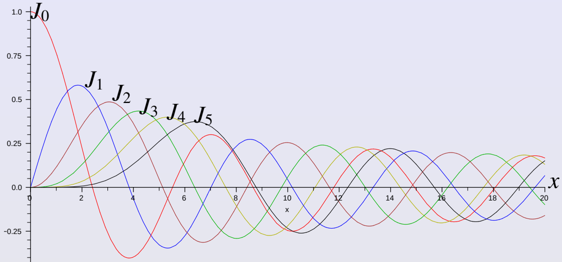
- Thus for each $m \in \mathbb{Z}$, $m \geq 0$ we have $J_m(\omega R) = 0$
- Thus the solution to the wave equation $\frac{\partial \psi}{\partial t^2} + \triangle_{\text{disc}} \psi = 0$ is a sum of solutions of the form

$$\psi(t, \rho, \theta) = e^{i\omega t} e^{im\theta} J_m(\omega \rho) \qquad \text{where} \qquad J_m(\omega R) = 0$$

- Therefore we need to know the zeros (roots) of the Bessel functions.
- If $\chi_m^s$ is a root of the Bessel function, i.e. $J_m(\chi_m^s) = 0$ then $\omega = \dfrac{\chi_m^s}{R}$ is a permitted value of $\omega$.

# Roots of Bessel functions

- Enumerate all the roots of $J_m$. That is let $\chi_m^s$ be the $s$'th root of $J_m$ so that
$$J_m(\chi_m^s) = 0$$
- Since for $m > 0$, $J_m(0) = 0$, we set $\chi_m^0 = 0$.
- Thus in all cases we let $\chi_m^1$ be the first positive root of $J_m$.




| | $\chi_m^0$ | $\chi_m^1$ | $\chi_m^2$ | $\chi_m^3$ | $\chi_m^4$ | $\chi_m^5$ | $\chi_m^6$ | $\chi_m^7$ |
|---|---|---|---|---|---|---|---|---|
| $J_0$ | – | 2.40 | 5.52 | 8.65 | 11.79 | 14.93 | 18.07 | 21.21 |
| $J_1$ | 0. | 3.83 | 7.01 | 10.17 | 13.32 | 16.47 | 19.62 | 22.76 |
| $J_2$ | 0. | 5.13 | 8.41 | 11.62 | 14.80 | 17.96 | 21.12 | 24.27 |
| $J_3$ | 0. | 6.38 | 9.76 | 13.02 | 16.22 | 19.41 | 22.58 | 25.75 |
| $J_4$ | 0. | 7.58 | 11.06 | 14.37 | 17.62 | 20.83 | 24.02 | 27.20 |
| $J_5$ | 0. | 8.77 | 12.34 | 15.70 | 18.98 | 22.22 | 25.43 | 28.63 |

# Complete solution to the wave equation on the disc

- Thus the complete solution to the wave equation for $\psi = \psi(t, \rho, \theta)$

$$\frac{\partial \psi}{\partial t^2} + \triangle_{\text{disc}} \psi = 0 \qquad \text{with boundary conditions} \qquad \psi(t, R, \theta) = 0$$

  is given by

$$\psi(t, \rho, \theta) = \sum_{s=1}^{\infty} \sum_{m=-\infty}^{\infty} \left( A_{ms} \exp\left(\frac{i\chi_m^s t}{R} + im\theta\right) J_m\left(\frac{\chi_m^s \rho}{R}\right) + B_{ms} \exp\left(\frac{-i\chi_m^s t}{R} + im\theta\right) J_m\left(\frac{\chi_m^s \rho}{R}\right)\right)$$

- The values of $A_{ms}, B_{ms}$ are determined by the initial conditions, via the use of the inner product.
- It is also posible to write the general solution in thers of cos and sin, so as to guarantee $\psi$ is real.

# Eigen properties of the disc harmonics

- The set of solutions to the $(\rho, \theta)$ part of the wave equation i.e. $\triangle_{\text{disc}}\mathcal{J} = -\omega^2\mathcal{J}$ subject to the boundary condition $\mathcal{J}(R, \theta) = 0$ are called **disc harmonics**.
- They are defined for each $m \in \mathbb{Z}$ and $s = 1, 2, \ldots$ I.e. we set

$$\mathcal{J}_m^s(\rho, \theta) = e^{im\theta} J_m\left(\frac{\chi_m^s \rho}{R}\right)$$

- The disc harmonics are eigenfunctions of $\triangle_{\text{disc}}$ and and the operator $\frac{\partial^2}{\partial^2\theta}$, with eigenvalues $-\left(\frac{\chi_m^s}{R}\right)^2$ and $-m^2$ respectively
- i.e.

$$\triangle_{\text{disc}}\mathcal{J}_m^s = -\left(\frac{\chi_m^s}{R}\right)^2 \mathcal{J}_m^s \qquad \text{and} \qquad \frac{\partial^2}{\partial\theta^2}\mathcal{J}_m^s = -m^2\mathcal{J}_m^s$$

# Inner product and initial conditions

- There is an inner product for functions on the disk given by

$$\langle f\,,\,g\rangle = \int_{\rho=0}^{R}\int_{\theta=0}^{2\pi} \rho f(\rho,\theta) g(\rho,\theta) d\theta\, d\rho$$

- Again the disc harmonics are orthogonal i.e.

$$\left\langle \mathcal{J}_m^s\,,\,\mathcal{J}_{m'}^{s'}\right\rangle = 0$$

If $m \neq m'$ or $n \neq n'$.

- However the disc harmonics are not normalised and

$$\langle \mathcal{J}_m^s\,,\,\mathcal{J}_m^s\rangle = \frac{\pi R^2}{2}\left(\frac{dJ_m}{d\rho}(\chi_m^s)\right)^2$$

- We use this inner product to calculate the coefficient $A_{ms}, B_{ms}$ in terms of the initial positions and velocities of the disc.

# Cylindrical wave guides

- Let us solve the equation for $\psi(t, \rho, \theta, z)$

$$\frac{\partial^2 \psi}{\partial t^2} - \triangle \psi = 0$$

subject to the boundary conditions

$$\psi(t, R, \theta, z) = 0 \qquad \text{and} \qquad \psi(t, 0, \theta, z) \quad \text{is bounded}$$

- By separation of variables $\psi(t, \rho, \theta) = T(t)\Upsilon(\rho, \theta)Z(z)$ we have

$$\frac{d^2 T}{dt^2} = -\omega^2 T\,, \qquad \triangle_{\text{disc}} \Upsilon = -k^2 \Upsilon \qquad \text{and} \qquad \frac{d^2 Z}{dz^2} = \epsilon \mu^2 Z$$

with

$$k = \pm \chi_m^s / R\,, \qquad \epsilon = \pm 1 \qquad \text{and} \qquad \omega^2 = k^2 - \epsilon \mu^2$$

- The sign of $\epsilon$ determines if the wave are propagating or non propagating.
  - If $\epsilon = 1$ the the waves are non propagating.
  - If $\epsilon = -1$ the the waves are propagating

# Non propagating waves

- If $\epsilon = 1$ then

$$\frac{d^2Z}{dz^2} = \mu^2 Z \quad \text{implies} \quad Z(z) = e^{\pm\mu z}$$

- We consider a semi infinite cylinder. I.e. $z > 0$.
- On “physical” grounds we exclude the solution $Z(z) \to \infty$ as $z \to \infty$. Thus we have $Z(z) = e^{-\mu z}$.
- This is a non propagating since $\psi \to 0$ exponentially as $z \to \infty$.

# Propagating waves

- If $\epsilon = -1$ then
  $$\frac{d^2Z}{dz^2} = -\mu^2 Z \quad \text{implies} \quad Z(z) = e^{\pm i\mu z} \quad \text{and} \quad \omega^2 = k^2 + \mu^2 = (\chi^s/R)^2 + \mu^2$$
- Again we consider a semi infinite cylinder. I.e. $z > 0$.
- $Z(z)$ remains order 1 as $z \to \infty$. So these are propagating waves.
- Solutions to the wave equation take the form
  $$\psi(t, \rho, \theta) = e^{\pm i\omega t \pm i\mu z + im\theta} J_m\left(\frac{\chi_m^s \rho}{R}\right)$$
- Observe that
  $$|\omega| \geq \chi^s/R \geq \chi_0^1/R \approx 2.4/R$$
  Thus there is a minimum frequency for propagating modes.
- The speed of propagations is given by
  $$\frac{|\mu|}{|\omega|} = \frac{\sqrt{\omega^2 - k^2}}{|\omega|} = \sqrt{1 - \frac{k^2}{\omega^2}} < 1$$
  Thus waves slow down in a cylinder. Without a cylinder $\frac{|\mu|}{|\omega|} = 1$.

# SECTION 3

## Wave Equation on the sphere and Spherical Harmonics

- The Laplace operator in spherical coordinates.
- The spherical wave equation.
- Separation of variables and the time equation.
- The spherical equation.
- The associated Legendre equation
- Eigen properties of spherical harmonics
- Orthogonality properties of spherical harmonics
- Laplace's equation and monopoles and dipoles.

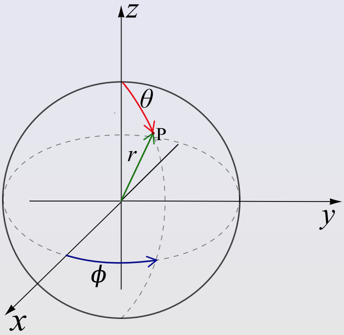
# Spherical Coordinates: Grad, Div and Laplacian

- Spherical Coordinates are given by $(r, \theta, \phi)$ where

$$r \geq 0\,, \qquad 0 \leq \theta \leq \pi \qquad \text{and} \qquad 0 \leq \phi \leq 2\pi$$

- The relationship with Cartesian coordinates $(x, y, z)$ is given by

$$x = r\sin\theta\cos\phi\,, \qquad y = r\sin\theta\sin\phi \qquad \text{and} \qquad z = r\cos\theta$$



# Spherical Coordinates

The orthogonal vector basis is given by

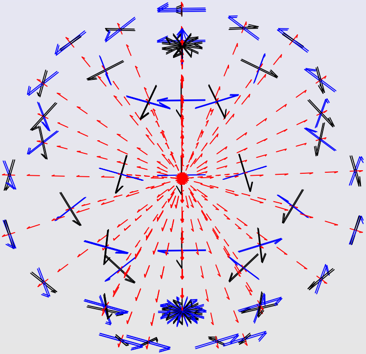
$$\underline{e}_r = \frac{\nabla r}{\|\nabla r\|}, \qquad \underline{e}_\theta = \frac{\nabla \theta}{\|\nabla \theta\|} \qquad \text{and} \qquad \underline{e}_\phi = \frac{\nabla \phi}{\|\nabla \phi\|}$$

Hence

$$\underline{e}_r = \sin\theta\cos\phi\,\underline{i} + \sin\theta\sin\phi\,\underline{j} + \cos\theta\,\underline{k},$$

$$\underline{e}_\theta = \cos\theta\cos\phi\,\underline{i} + \cos\theta\sin\phi\,\underline{j} + \sin\theta\,\underline{k} \qquad \text{and}$$

$$\underline{e}_\phi = -\sin\phi\,\underline{i} + \cos\phi\,\underline{j}$$



$\underline{e}_r$ are red

$\underline{e}_\phi$ are blue

$\underline{e}_\theta$ are black

# Spherical Coordinates: Grad, Div and Laplacian

- Let $\psi = \psi(r, \theta, z)$ and $\underline{v} = v_r \underline{e}_r + v_\theta \underline{e}_\theta + v_z \underline{e}_z$, where $v_r = v_r(r, \theta, z)$ etc.
- The Grad, Div and Laplacian are given by

$$\nabla\psi = \frac{\partial\psi}{\partial r}\underline{e}_r + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\underline{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}\underline{e}_z$$

$$\nabla\cdot\underline{v} = \frac{1}{r}\frac{\partial}{\partial r^2}\left(r^2 v_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta v_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}$$

$$\triangle\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin\theta}\frac{\partial^2\psi}{\partial\phi^2}$$

- The curl is given by

$$\nabla\times\underline{v} = \frac{1}{r\sin\theta}\left(\frac{\partial\sin\theta v_\phi}{\partial\theta} - \frac{\partial v_\theta}{\partial\phi}\right)\underline{e}_r + \left(\frac{1}{r\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{1}{r}\frac{\partial}{\partial r}\left(rv_\phi\right)\right)\underline{e}_\theta + \frac{1}{r}\left(\frac{\partial\left(rv_\theta\right)}{\partial r} - \frac{\partial v_r}{\partial\theta}\right)\underline{e}_\phi$$

# The Laplace operator in spherical coordinates.

- Recall the Laplace operator in spherical coordinates.

$$\triangle\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}$$

- We can rewrite this as

$$\triangle\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2}\triangle_{\text{sph}}\psi$$

where

$$\triangle_{\text{sph}}\psi = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}$$

is the Laplace operator on the sphere.

- Laplace's equation:

$$\triangle\psi = 0 \qquad \text{for} \qquad \psi = \psi(r,\theta,\phi)$$

$\Psi$ could be the gravitational potential or electrostatic potential for space with a single source at the origin $r = 0$.
We shall solve this later.

# The spherical wave equation.

- First we shall solve the spherical wave equation

$$\frac{\partial^2 \psi}{\partial t^2} - \triangle_{\mathrm{sph}} \psi = 0 \qquad \text{for} \qquad \psi = \psi(t, \theta, \phi)$$

where

$$\triangle_{\mathrm{sph}} \psi = \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \psi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \psi}{\partial \phi^2}$$

- We may think of a rubber ball, with air inside. Let $\psi$ be the radial displacement of the ball from the rest spherical position.
- For small displacements and a short period of time, the spherical wave equation is a reasonable model for the motion of the surface of the ball.
- However it is linear, so is a bad approximation for large displacements, and there is no damping term so the motion will continue for ever.
- Particluar solutions are called **spherical harmonics**.

# Separation of variables.

- We wish to solve the spherical wave equation

$$\frac{\partial^2 \psi}{\partial t^2} - \triangle_{\text{sph}} \psi = 0 \qquad \text{for} \qquad \psi = \psi(t, \theta, \phi)$$

- We can totally separate the spherical wave equation. However we shall separate $t$ first, then $(\theta, \phi)$.
- Thus let $\psi(t, \theta, \phi) = T(t)\,\Upsilon(\theta, \phi)$.
- Show this gives

$$\frac{d^2 T}{dt^2} + \omega^2 T = 0 \qquad \text{and} \qquad \triangle_{\text{sph}} \Upsilon + \omega^2 \Upsilon = 0$$

  where $-\omega^2$ is the constant of separation.
- The $T$ equation is therefore solved for fixed frequency by setting

$$T(t) = A_\omega \cos(\omega t) + B_\omega \sin(\omega t)$$

# The spherical equation

- For $\Upsilon = \Upsilon(\theta, \phi)$ we have

$$\triangle_{\text{sph}} \Upsilon + \omega^2 \Upsilon = 0$$

- We set $\omega^2 = n(n+1)$. (We see whay later.)
- We shall see that the solutions to the spherical equation require that $n$ is an integer. What are the frequencies?
- Thus we have

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Upsilon}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2\Upsilon}{\partial\phi^2} + n(n+1)\Upsilon = 0$$

- We again use separation of variables, and set $\Upsilon(\theta, \phi) = \Theta(\theta)\Phi(\phi)$. Show this gives

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \left(n(n+1) - \frac{m^2}{\sin^2\theta}\right)\Theta = 0$$

and

$$\frac{d^2\Phi}{d\phi^2} + m^2\Phi = 0$$

# The $\Phi(\phi)$ equation

- $$\frac{d^2\Phi}{d\phi^2} + m^2\Phi = 0$$

  This is clearly simple harmonic oscillator.
- However the boundary condition are now periodic. Since $\phi$ is an angle. Thus $\Phi(\phi) = \Phi(\phi + 2\pi)$.
- We use the complex periodic functions. Thus we set

$$\Phi(\phi) = e^{im\phi}$$

- For this to be periodic, we require that $m \in \mathbb{Z}$.
- For real solutions of the wave equation we must take the real component

$$\psi(t, \theta, \phi) = \mathrm{Re}(T(t)\Theta(\theta)\Phi(\phi))$$

# The associated Legendre equation

- $$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+\left(n(n+1)-\frac{m^2}{\sin^2\theta}\right)\Theta=0$$
- We substitute in $z=\cos\theta$, which coincidental it corresponds to the $z$ coordinated. We set
$$P(z)=P(\cos\theta)=\Theta(\theta)$$
Thus differentiating both sides by $\theta$ gives
$$-\sin\theta\frac{dP}{dz}(\cos\theta)=\frac{d\Theta}{d\theta}$$
Thus
$$-\sin^2\theta\frac{dP}{dz}(\cos\theta)=\sin\theta\frac{d\Theta}{d\theta}$$
- Differentiating again by $\theta$ gives
$$\sin\theta\frac{d}{dz}\left(\sin^2\theta\frac{dP}{dz}(\cos\theta)\right)=\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)$$

# The associated Legendre equation

- Hence

$$\frac{d}{dz}\left(\sin^2\theta\frac{dP}{dz}(\cos\theta)\right)=\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)$$

- Substituting this into our original $\Theta$ equation

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+\left(n(n+1)-\frac{m^2}{\sin^2\theta}\right)\Theta=0$$

  we have

$$\frac{d}{dz}\left(\sin^2\theta\frac{dP}{dz}(\cos\theta)\right)+\left(n(n+1)-\frac{m^2}{\sin^2\theta}\right)P=0$$

- Since $z=\cos\theta$ and $\sin^2\theta=1-\cos^2\theta=1-z^2$. Thus

$$\frac{d}{dz}\left((1-z^2)\frac{dP}{dz}\right)+\left(n(n+1)-\frac{m^2}{1-z^2}\right)P=0$$

  This is the **Associated Legendre Equation**.

# The associated Legendre polynomials.

- Since $P$ clearly depends on $n$ and $m$ as well as $z$ we write $P_n^m(z)$. Thus we have the associated Legendre equation

$$\frac{d}{dz}\left((1-z^2)\frac{dP_n^m}{dz}\right) + \left(n(n+1) - \frac{m^2}{1-z^2}\right)P_n^m = 0$$

- The associated Legendre equation is a second order linear ODE. Therefore there are two solutions.
- If

$$n \in \{0, 1, 2, 3, \dots\} \qquad \text{and} \qquad m = -n, -n+1, \dots, n-1, n$$

  then there is one solution which is bounded. This is the **associated Legendre polynomial**.

$$P_n^m(z) = \frac{1}{2^n n!}(1-z^2)^{m/2}\,\frac{d^{n+m}}{dz^{n+m}}(z^2-1)^n.$$

- If $n, m$ are not in these ranges then there is no bounded solution.
- The constant $1/2^n n!$ is simply convention.
- Observe that if $m$ is odd then $P_n^m$ is not a polynomial, but the name has stuck.

# Spherical harmonics summary

- Up to $\pm 1$ the spherical harmonics are therefore given by

$$Y_n^m(\theta, \phi) = \left( \frac{2n+1}{4\pi} \frac{(n-m)!}{(n+m)!} \right)^{1/2} e^{im\phi} P_n^m(\cos\theta)$$

  where

$$P_n^m(z) = \frac{1}{2^n n!}(1-z^2)^{m/2} \frac{d^{n+m}}{dz^{n+m}}(z^2-1)^n.$$

- The solution to the wave equation for $\psi = \psi(t, \theta, \phi)$

$$\frac{\partial^2 \psi}{\partial \psi^2} - \triangle_{\mathrm{sph}} \psi = 0$$

  is given by

$$\psi = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left( A_{nm} \cos\left(\sqrt{n(n+1)}t\right) + B_{nm} \sin\left(\sqrt{n(n+1)}t\right) \right) Y_n^m(\theta, \phi)$$

# Some spherical harmonics

$$Y_0^0(\theta,\phi) = \frac{1}{2}\sqrt{\frac{1}{\pi}}$$

$$Y_1^{-1}(\theta,\phi) = \frac{1}{2}\sqrt{\frac{3}{2\pi}}\, e^{-i\phi}\ \sin\theta$$

$$Y_1^{0}(\theta,\phi) = \frac{1}{2}\sqrt{\frac{3}{\pi}}\ \cos\theta$$

$$Y_1^{1}(\theta,\phi) = \frac{-1}{2}\sqrt{\frac{3}{2\pi}}\, e^{i\phi}\ \sin\theta$$

$$Y_2^{-2}(\theta,\phi) = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\, e^{-2i\phi}\ \sin^2\theta$$

$$Y_2^{-1}(\theta,\phi) = \frac{1}{2}\sqrt{\frac{15}{2\pi}}\, e^{-i\phi}\ \sin\theta\ \cos\theta$$

$$Y_2^{0}(\theta,\phi) = \frac{1}{4}\sqrt{\frac{5}{\pi}}\,(3\cos^2\theta - 1)$$

$$Y_2^{1}(\theta,\phi) = \frac{-1}{2}\sqrt{\frac{15}{2\pi}}\, e^{i\phi}\ \sin\theta\ \cos\theta$$

$$Y_2^{2}(\theta,\phi) = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\, e^{2i\phi}\ \sin^2\theta$$

# Eigen properties of spherical harmonics

- By construction the spherical harmonics $Y_n^m(\theta, \phi)$ are eigenfunctions of the spherical Laplacian

$$\triangle_{\mathrm{sph}} Y_n^m = -n(n+1) Y_n^m$$

- Also since

$$Y_n^m = (-1)^m \left( \frac{2n+1}{4\pi} \frac{(n-m)!}{(n+m)!} \right)^{1/2} e^{im\phi} P_n^m(\cos\theta)$$

  we see that

$$\frac{\partial}{\partial \phi} Y_n^m = im Y_n^m$$

- Thus the spherical harmonics $Y_n^m(\theta, \phi)$ are simultaneously eigenfunctions of two linear operators.
- A particular spherical harmonic $Y_n^m(\theta, \phi)$ is determined by the numbers $n$ and $m$. This is equivalent to stating the eigenvalues of the operators $\triangle_{\mathrm{sph}}$ and $\frac{\partial}{\partial \phi}$.
- Since the spherical harmonics simultaneously "diagonalise" the operators $\triangle_{\mathrm{sph}}$ and $\frac{\partial}{\partial \phi}$. This implies that these two operators must commute.

# Inner product on spherical functions

- A general complex valued function on the sphere is written $f : S^2 \to \mathbb{C}$. This means, in spherical coordinates $(\theta, \phi)$ we can write $f = f(\theta, \phi)$ where $f(\theta, \phi) \in \mathbb{C}$.
- There is a natural inner product on functions on the sphere. Given two complex valued functions $f, g : S^2 \to \mathbb{C}$ on the sphere. We define the inner product of the functions $f$ and $g$ as

$$\langle f \, , \, g \rangle = \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \overline{f(\theta, \phi)} \, g(\theta, \phi) \, \sin\theta \, d\theta \, d\phi$$

- With respect to this inner product the spherical harmonics are orthonormal. That is

$$\left\langle Y_n^m \, , \, Y_{n'}^{m'} \right\rangle = \delta_{nn'} \delta_{mm'}$$

# Completeness properties of spherical harmonics

- Any continuous complex spherical function $f : S^2 \to \mathbb{C}$ can be expanded into a series

$$f = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} a_{nm} Y_n^m \qquad \text{i.e.} \qquad f(\theta, \phi) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} a_{nm} Y_n^m(\theta, \phi)$$

- From orthogonality we have

$$\left\langle Y_{n'}^{m'}, f \right\rangle = a_{n'm'} \qquad \text{Why?}$$

Hence

$$f = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \langle Y_n^m, f \rangle Y_n^m$$

- We use this property to calculate the coefficients $A_{nm}$ and $B_{nm}$ for the general solution to the wave equation from the initial conditions $\psi(0, \theta, \phi)$ and $\dot{\psi}(0, \theta, \phi)$.

# Laplace's equation for spherical symmetric space

- We can now solve Laplace's equation for $\psi(r, \theta, \phi)$

$$\triangle\psi = 0$$

  for spherical symmetric situations.
- There are two scenarios we can consider.
  - All of $\mathbb{R}^3$ space with the origin, $r = 0$, removed. This is because we put a point source at the origin.
    The Laplace's equation gives the gravitational potential for a point mass.
    The Laplace's equation also gives the electrostatic potential a monopole, dipole, etc.
    We therefore consider solutions where $\psi$ is bounded for large $r$ but is allowed to go infinite for small $r$.
  - Laplace's equation for a ball.
    This corresponds to the gravitational potential, inside a massive shell.
    Here we require that $\psi(r, \theta, \phi)$ is bounded at the origin, $r = 0$.

# Laplace's equation for spherical symmetric space

- Recall that Laplace's equation $\triangle\psi$ for $\psi = \psi(r, \theta, \phi)$ is given by

$$\triangle\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2} = 0$$

- We can rewrite this as

$$\triangle\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2}\triangle_{\text{sph}}\psi = 0$$

- And $\triangle_{\text{sph}} Y_n^m = -n(n+1)Y_n^m$ where $Y_n^m(\theta, \phi)$ is the spherical harmonic.
- Use separation of variables. Set $\psi(r, \theta, \phi) = R(r)\Upsilon(\theta, \phi)$. Hence the radial equation is given by

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = n(n+1)R$$

where $n = 0, 1, 2, \ldots$. Show

# The radial equation

- $$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = n(n+1)R$$
- We substitute in the ansatz $R(r) = r^\alpha$. Show this implies
$$\alpha(\alpha+1) = n(n+1)$$
- This can be satisfied by setting
$$\alpha = n \qquad \text{and} \qquad \alpha = -n-1$$
- The solution to Laplace's equation $\triangle\psi = 0$ in spherical coordinates is
$$\psi = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\left(A_{nm}r^n + B_{nm}r^{-n-1}\right)Y_n^m(\theta,\phi)$$

# Monopoles and dipoles

- Let us consider solutions where $\psi$ is bounded as $r \to \infty$. Thus $A_{nm} = 0$ and

$$\sum_{n=0}^{\infty} \sum_{m=-n}^{n} B_{nm} r^{-n-1} Y_n^m(\theta, \phi)$$

- Thus the first few terms are

$$\psi = \frac{B_{0,0}}{r} + \frac{1}{r^2}\left(B_{1,-1}Y_1^{-1} + B_{1,0}Y_1^0 + B_{1,1}Y_1^1\right) + \frac{1}{r^3}\left(B_{2,-2}Y_2^{-2} + B_{2,-1}Y_2^{-1} + B_{2,0}Y_2^0 + B_{2,1}Y_2^1 + B_{2,2}Y_2^2\right)$$

- The first term is due to a monopole. This is spherically symmetric.
- The next three terms are due to simple dipoles. Any possible direction dipole can be constructed by setting $B_{1,-1}, B_{1,0}, B_{1,1}$. Observe these fall off quicker. Therefore dipoles are weaker than monopoles for large distances.
- The next five terms are due to a quadrupole. These fall off even quicker $1/r^3$.

# Inside a ball

- Let us here consider only solutions which are bounded at $r = 0$. Hence all the $B_{nm} = 0$ and

$$\psi = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} A_{nm} r^n Y_n^m(\theta, \phi)$$

- The first few terms are

$$\begin{aligned} \psi = A_{0,0} &+ r\left(A_{1,-1}Y_1^{-1} + A_{1,0}Y_1^0 + A_{1,1}Y_1^1\right) \\ &+ r^2\left(A_{2,-2}Y_2^{-2} + A_{2,-1}Y_2^{-1} + A_{2,0}Y_2^0 + A_{2,1}Y_2^1 + A_{2,2}Y_2^2\right) \end{aligned}$$

- The first term gives us the spherically symmetric potential inside a ball. This is a constant potential.
- The next terms give us non a symmetric potential.