Introduction to hydrodynamical descriptions of plasmas

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Introduction

Plasmas: ionized gas with suitably high number density of free electrons and ions exhibiting "collective" behaviour

- More appropriate to describe system *as a whole* (macroscopic) rather than as individual particles (microscopic)
- Regard as an "electron fluid" mixed with an "ion fluid"

Introduction

- Term "plasma" introduced by

Irving Langmuir (1881-1957)

- Reminded him of blood plasma (liquid component of blood in which cells are suspended)

Contents

Books :

Lautrup, "Physics of Continuous Matter"

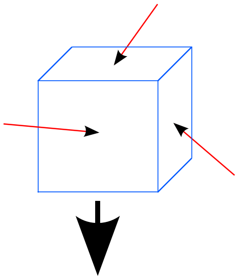
Raichaudhuri, "The Physics of Fluids and Plasmas"

Part 1 : Fluids

Static fluids

Static fluids

Forces on an infinitesimal element of *static* fluid

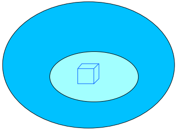


- Red arrows are contact forces (short distance) acting across the element's faces (due to adjacent elements)
- Large black arrow is the body force (long distance) acting on the element (e.g. gravitational field, electric field)
- Total force on element must vanish (element at rest)

Total force

$$d\mathbf{F}_{\text{body}} + \sum_{\text{faces } \mathbf{n}} d\mathbf{F}^{\mathbf{n}}_{\text{contact}} = 0 \qquad (1)$$

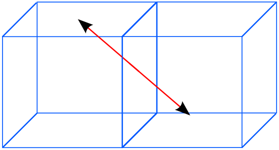
where **n** is the outward pointing unit normal to the face.
Sum over all elements of a volume of fluid



$$\sum_{\text{elements}} d\mathbf{F}_{\text{body}} + \sum_{\text{elements}} \left(\sum_{\text{faces } \mathbf{n}} d\mathbf{F}^{\mathbf{n}}_{\text{contact}} \right) = 0 \qquad (2)$$

Newton's 3rd law

Newton's 3rd law : action and reaction are equal and opposite



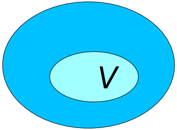
Contact forces at adjoining faces cancel

$$\implies \sum_{\text{elements}} d\mathbf{F}_{\text{body}} + \sum_{\text{open faces } \mathbf{n}} d\mathbf{F}^{\mathbf{n}}_{\text{contact}} = 0 \qquad (3)$$

Continuum limit

Taking the limit of the number of elements $\to \infty$ it follows

$$\int_V d\mathbf{F}_{\text{body}} + \int_{\partial V} d\mathbf{F}^{\mathbf{n}}_{\text{contact}} = 0 \tag{4}$$



Holds for any control volume V in the fluid

Examples of body force

- Gravitational force

$$dF_{\text{body}} = \rho_{\text{mass}} \mathbf{g} \, dV \tag{5}$$

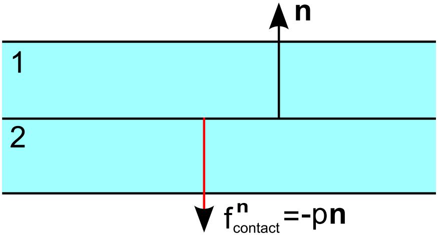
where $\mathbf{g} = -9.81\text{ms}^{-2}\,\mathbf{z}$ and ρ_{mass} is the fluid's mass density

- Electric force

$$d\mathbf{F}_{\text{body}} = \rho_{\text{charge}} \mathbf{E} \, dV \tag{6}$$

where $\mathbf{E}$ is a time-independent electric field and ρ_{charge} is the fluid's charge density

Contact force



Contact force per unit area $\mathbf{f}^{\mathbf{n}}_{\mathrm{contact}}$ exerted by plane 1 on plane 2 must be parallel to the unit normal $\mathbf{n}$

- component perpendicular to $\mathbf{n}$ would lead to motion

Contact force

The infinitesimal contact force $d\mathbf{F}^{\mathbf{n}}_{\text{contact}}$ acting at an infinitesimal surface with surface element dS is

$$d\mathbf{F}^{\mathbf{n}}_{\text{contact}} = \mathbf{f}^{\mathbf{n}}_{\text{contact}} dS = -p\mathbf{n} dS \tag{7}$$

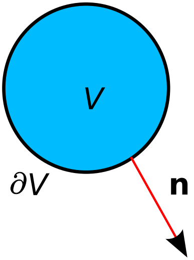
where $\mathbf{n}$ is the unit normal to dS pointing towards the fluid matter exerting the contact force

- the scalar field p is the hydrostatic pressure

Global equation of hydrostatic balance

$$\boxed{\int_V \mathbf{f}_{\text{body}} dV - \int_{\partial V} p\mathbf{n} dS = 0}$$

where $\mathbf{f}_{\text{body}}$ is the body force density (body force per unit volume)



Local equation of hydrostatic balance

1D fluid

$$\int_{x_1}^{x_2} f_{\text{body}}\,dx - \big(p(x_2) - p(x_1)\big) = 0 \tag{8}$$

$$\implies \int_{x_1}^{x_2} \Big(f_{\text{body}} - \frac{dp}{dx}\Big)dx = 0 \tag{9}$$

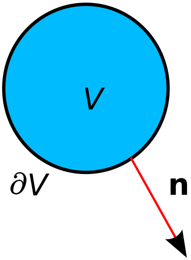
True for all control intervals so

$$\boxed{f_{\text{body}} - \frac{dp}{dx} = 0}$$

3D considerations

Divergence theorem :

$$\int_{\partial V} \mathbf{u} \cdot \mathbf{n} \, dS = \int_V \boldsymbol{\nabla} \cdot \mathbf{u} \, dV \tag{10}$$



$$\implies \int_{\partial V} p\mathbf{x} \cdot \mathbf{n} \, dS = \int_V \boldsymbol{\nabla} \cdot (p\mathbf{x}) \, dV = \int_V \frac{\partial p}{\partial x} dV \tag{11}$$

3D considerations

Hence

$$\begin{aligned}\int_{\partial V} p\mathbf{n}\,dS =& \left(\int_{\partial V} p\mathbf{x}\cdot\mathbf{n}\,dS\right)\mathbf{x} \\ &+ \left(\int_{\partial V} p\mathbf{y}\cdot\mathbf{n}\,dS\right)\mathbf{y} + \left(\int_{\partial V} p\mathbf{z}\cdot\mathbf{n}\,dS\right)\mathbf{z} \\ =& \left(\int_V \frac{\partial p}{\partial x}\,dV\right)\mathbf{x} \\ &+ \left(\int_V \frac{\partial p}{\partial y}\,dV\right)\mathbf{y} + \left(\int_V \frac{\partial p}{\partial z}\,dV\right)\mathbf{z} \\ =& \int_V \boldsymbol{\nabla} p\,dV \qquad (12)\end{aligned}$$

Local equation of hydrostatic balance

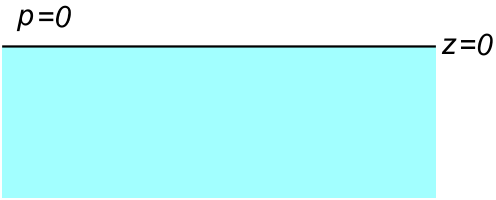
3D fluid

$$\int_V (\mathbf{f}_{\text{body}} - \boldsymbol{\nabla} p) dV = 0 \tag{13}$$

is true for all V so

$$\boxed{\mathbf{f}_{\text{body}} - \boldsymbol{\nabla} p = 0}$$

Example : Pressure vs depth



Body force density due to gravity on a fluid with constant mass density $\rho_{\text{mass}} = \rho_0$

$$\mathbf{f}_{\text{body}} = -\rho_0\, g_0 \mathbf{z} \tag{14}$$

where $g_0 = 9.81\text{ms}^{-2}$

Example : Pressure vs depth

Local equation of hydrostatic equilibrium in a body of water at rest

$$-\rho_0 g_0 \mathbf{z} - \nabla p = 0 \tag{15}$$

hence

$$\frac{\partial p}{\partial x} = 0, \qquad \frac{\partial p}{\partial y} = 0, \qquad \frac{\partial p}{\partial z} = -\rho_0 g_0$$

$$\text{and } p = 0 \text{ at } z = 0$$

$$\implies p(z) = -\rho_0 g_0 z \tag{16}$$

i.e. $p = \rho_0 g_0 h$ where $h = -z$ is the depth

Example : Self-gravitating ball of fluid (a star)

- Radius of star is R
- Mass density $\rho_{\text{mass}} = \rho_0$ is constant

Newton's law of gravitation :

$$\nabla^2 \Phi = 4\pi G \, \rho_0 \tag{17}$$

where $\mathbf{g} = -\nabla \Phi$.
Local equation of hydrostatic balance

$$-\rho_0 \nabla \Phi - \nabla p = 0 \tag{18}$$

so p satisfies

$$\nabla^2 p = -4\pi G \, \rho_0^2 \text{ with } p = 0 \text{ at } r = R \tag{19}$$

Example : Self-gravitating ball of fluid (a star)

Spherical symmetric pressure

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dp}{dr}\right) = -4\pi G\,\rho_0^2$$

$$\implies p = -\frac{2\pi G}{3}\rho_0^2 r^2 + \frac{a}{r} + b \qquad (20)$$

where a and b are constants of integration.

- Boundary conditions :
  Pressure is finite at $r = 0$ and pressure vanishes at $r = R$

$$\implies p = \frac{2\pi G}{3}\rho_0^2(R^2 - r^2) \qquad (21)$$