

BASIC MATHEMATICAL FORMULAE

Vectors

For vectors

$$\begin{aligned}\mathbf{u} &= (u_x, u_y, u_z) = u_x\mathbf{i} + u_y\mathbf{j} + u_z\mathbf{k} \\ \mathbf{v} &= (v_x, v_y, v_z) = v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k},\end{aligned}$$

where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are unit vectors along Cartesian axes, the *scalar* or *dot product* is

$$\mathbf{u} \cdot \mathbf{v} = u_x v_x + u_y v_y + u_z v_z \quad (1)$$

and the length of a vector is

$$|u| = \sqrt{\mathbf{u} \cdot \mathbf{u}} = \sqrt{u_x^2 + u_y^2 + u_z^2} \quad (2)$$

The *vector* or *cross product* is

$$\mathbf{u} \wedge \mathbf{v} = \mathbf{u} \times \mathbf{v} = (u_y v_z - u_z v_y)\mathbf{i} + (u_z v_x - u_x v_z)\mathbf{j} + (u_x v_y - u_y v_x)\mathbf{k}. \quad (3)$$

This can also be written in determinant form as

$$\mathbf{u} \wedge \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{vmatrix} \quad (4)$$

Standard identities:

$$\mathbf{u} \cdot \mathbf{v} \wedge \mathbf{w} = \mathbf{v} \cdot \mathbf{w} \wedge \mathbf{u} = \mathbf{w} \cdot \mathbf{u} \wedge \mathbf{v} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix} \quad (5)$$

$$\mathbf{u} \wedge (\mathbf{v} \wedge \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w} \quad (6)$$

Vector Calculus

The *gradient* of a scalar ϕ is

$$\text{grad } \phi = \nabla \phi = \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right). \quad (7)$$

The *divergence* of a vector \mathbf{u} is

$$\text{div } \mathbf{u} = \nabla \cdot \mathbf{u} = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}, \quad (8)$$

and the *curl* or *rotation* of the vector is

$$\text{curl } \mathbf{u} = \text{rot } \mathbf{u} = \nabla \wedge \mathbf{u} = \left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z} \right) \mathbf{i} + \left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x} \right) \mathbf{j} + \left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y} \right) \mathbf{k}. \quad (9)$$

The *Laplacian* operator applied to a scalar ϕ is

$$\nabla^2 \phi = \nabla \cdot \nabla \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}, \quad (10)$$

as follows from (7) and (8).

In cylindrical coordinates (r, θ, z) ,

$$\nabla\phi = \left(\frac{\partial\phi}{\partial r}, \frac{1}{r} \frac{\partial\phi}{\partial\theta}, \frac{\partial\phi}{\partial z} \right), \quad (11)$$

$$\operatorname{div} \mathbf{u} = \nabla \cdot \mathbf{u} = \frac{1}{r} \frac{\partial}{\partial r} (ru_r) + \frac{1}{r} \frac{\partial u_\theta}{\partial\theta} + \frac{\partial u_z}{\partial z}, \quad (12)$$

$$\nabla^2\phi = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial\phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial z^2}. \quad (13)$$

Other basic results include

$$\nabla \wedge \nabla\phi = 0, \quad \nabla \cdot \nabla \wedge \mathbf{u} = 0. \quad (14)$$

Gauss' or Divergence Theorem

For a closed surface S containing a volume V ,

$$\iint_S \mathbf{a} \cdot d\mathbf{S} = \iiint_V \nabla \cdot \mathbf{a} \, dV \quad (15)$$

where $d\mathbf{S}$ is in the direction of the outward pointing normal. Related results are:

$$\iiint_V \nabla\phi \, dV = \iint_S \phi \, d\mathbf{S} \quad (16)$$

$$\iiint_V \nabla \wedge \mathbf{u} \, dV = - \iint_S \mathbf{u} \wedge d\mathbf{S}. \quad (17)$$

Stokes' Theorem

For an open surface S with an oriented boundary curve C ,

$$\oint_C \mathbf{a} \cdot d\mathbf{l} = \iint_S \nabla \wedge \mathbf{a} \cdot d\mathbf{S} \quad (18)$$

A related result is

$$\iint_S d\mathbf{S} \wedge \nabla\phi = \oint_C \phi \, d\mathbf{l}. \quad (19)$$

Vector Calculus Identities

$$\nabla(\psi + \phi) = \nabla\psi + \nabla\phi \quad (20)$$

$$\nabla(\phi\psi) = \phi\nabla\psi + \psi\nabla\phi \quad (21)$$

$$\nabla \cdot (\mathbf{u} + \mathbf{v}) = \nabla \cdot \mathbf{u} + \nabla \cdot \mathbf{v} \quad (22)$$

$$\nabla \cdot (\phi\mathbf{u}) = \phi\nabla \cdot \mathbf{u} + \mathbf{u} \cdot \nabla\phi \quad (23)$$

$$\nabla \wedge (\mathbf{u} + \mathbf{v}) = \nabla \wedge \mathbf{u} + \nabla \wedge \mathbf{v} \quad (24)$$

$$\nabla \wedge (\phi\mathbf{u}) = \phi\nabla \wedge \mathbf{u} + \nabla\phi \wedge \mathbf{u} \quad (25)$$

$$\nabla \cdot (\mathbf{u} \wedge \mathbf{v}) = \mathbf{v} \cdot \nabla \wedge \mathbf{u} - \mathbf{u} \cdot \nabla \wedge \mathbf{v} \quad (26)$$

$$\nabla \wedge (\nabla \wedge \mathbf{u}) = \nabla(\nabla \cdot \mathbf{u}) - \nabla^2\mathbf{u} \quad (27)$$

$$\nabla \cdot \nabla \wedge \mathbf{u} = 0 \quad (28)$$

$$\nabla \wedge \nabla\phi = 0 \quad (29)$$

Note that (27) defines ∇^2 operating on a vector. In Cartesian coordinates $\nabla^2 \mathbf{u} = (\nabla^2 u_x, \nabla^2 u_y, \nabla^2 u_z)$ but in other systems, $\nabla^2 \mathbf{u}$ is not in general the Laplacian operating on components separately.

Complex Numbers

With the notation $i = \sqrt{-1}$, an important result is

$$e^{i\theta} = \cos \theta + i \sin \theta. \quad (30)$$

We can write

$$i = e^{i\pi/2} \quad \text{and} \quad \sqrt{i} = i^{1/2} = e^{i\pi/4} = \cos(\frac{1}{4}\pi) + i \sin(\frac{1}{4}\pi) = \frac{1}{\sqrt{2}}(1 + i). \quad (31)$$

When dealing with time dependent quantities such as plane electromagnetic waves, it is common to use a complex form where the real part is understood. Thus, if $C = c_1 + ic_2$,

$$Ce^{i\omega t} \longleftrightarrow \text{Re} (Ce^{i\omega t}) = c_1 \cos \omega t - c_2 \sin \omega t.$$

Physical constants

Electron charge e	1.602177×10^{-19} coulomb
Electron mass m_e	9.108×10^{-31} kg
e/m_e	1.759×10^{11} coulomb kg ⁻¹
Proton mass m_p	1.672×10^{-27} kg
m_p/m_e	1836
Vacuum permeability μ_0	$4\pi \times 10^{-7}$ henry m ⁻¹
Vacuum speed of light c	2.998×10^8 m sec ⁻¹
Vacuum permittivity $\epsilon_0 = 1/(\mu_0 c^2)$	8.854×10^{-12} farad m ⁻¹
Vacuum impedance $Z_0 = (\mu_0/\epsilon_0)^{1/2}$	376.7 ohm
Vacuum admittance $Y_0 = 1/Z_0$	2.654×10^{-3} mho
Electron rest energy $m_e c^2$	8.186×10^{-14} joule (5.110×10^5 eV)
Classical electron radius $r_e = \mu_0 e^2 / (4\pi m_e)$	2.818×10^{-15} m
Planck's constant h	6.625×10^{-34} joule sec
$\hbar = h/(2\pi)$	1.054×10^{-34} joule sec
Fine structure constant $e^2/(2\epsilon_0 \hbar c)$	7.297×10^{-3}
Bohr radius $a_0 = 4\pi\epsilon_0 \hbar^2 / (m_e e^2)$	5.292×10^{-11} m
Bohr magneton $e\hbar/2m_e$	9.273×10^{-24} amp m ²
Boltzmann's constant K	1.380×10^{-23} joule deg ⁻¹