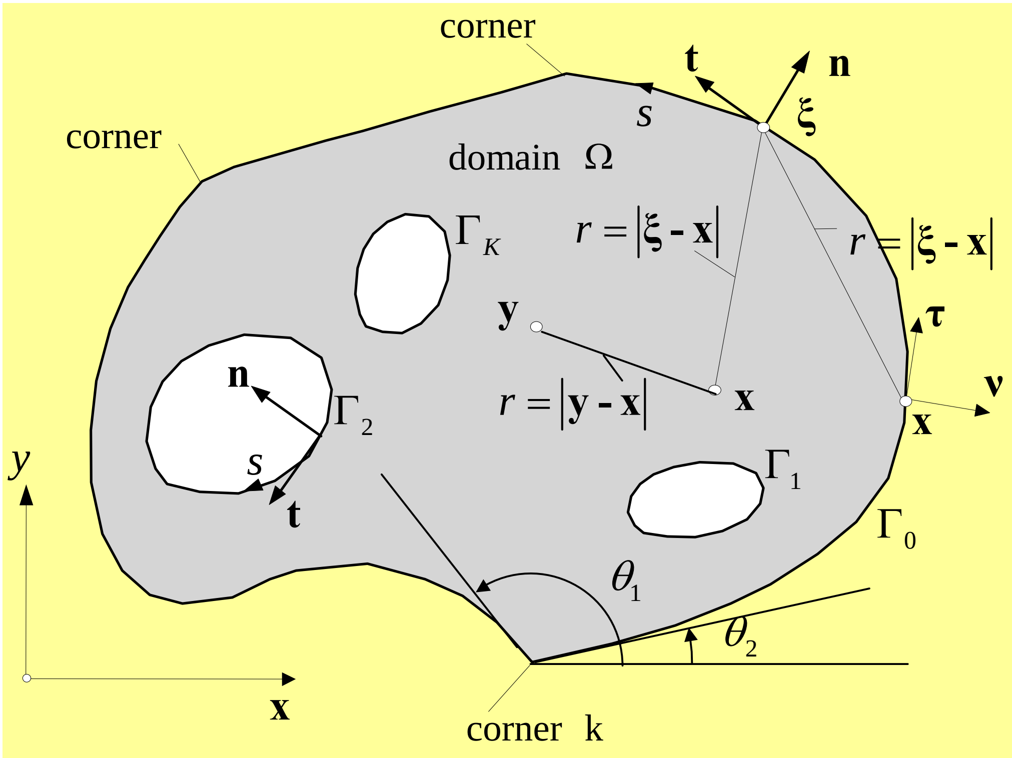
Part B

(continued)

The AEM for Elliptic PDEs

The AEM for 4nd order elliptic PDEs [20,21,22,23]



Geometry of the plate and notation

Equations governing the deflections of thin plates are the most common examples of fourth order elliptic PDEs. Without limiting the generality, we will demonstrate the method by applying it to the Bending problem of a plate with combined action of membrane forces.

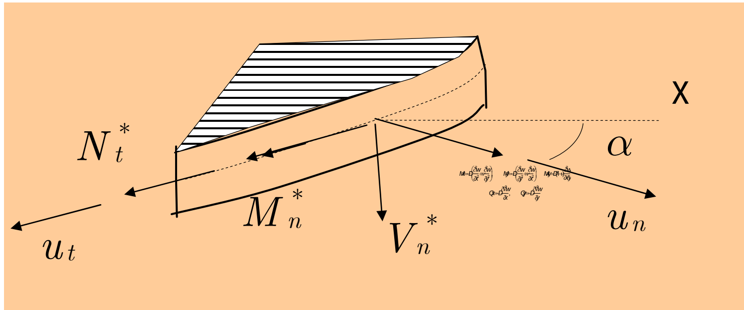
The PDE for plate bending under transverse and membrane forces

$$D\nabla^4 w - (N_x w_{,xx} + 2N_{xy} w_{,xy} + N_y w_{,yy}) + q_x w_{,x} + q_y w_{,y} = f \quad \text{in} \quad \Omega$$

The BCs

$$\left.\begin{aligned} Vw + N_n^* w_{,n} + N_t^* w_{,t} + k_T w = V_n^* \quad &\text{or} \quad w = w^* \\ Mw + k_R w_{,n} = M_n^* \quad &\text{or} \quad w_{,n} = w_{,n}^* \end{aligned}\right\} \text{ on } \Gamma$$

$$k_T^{(k)} w^{(k)} - [\![Tw]\!]_k = R_k^* \quad \text{or} \quad w^{(k)} = w_k^* \qquad \text{at corner point k}$$



w = deflection

f = transverse load

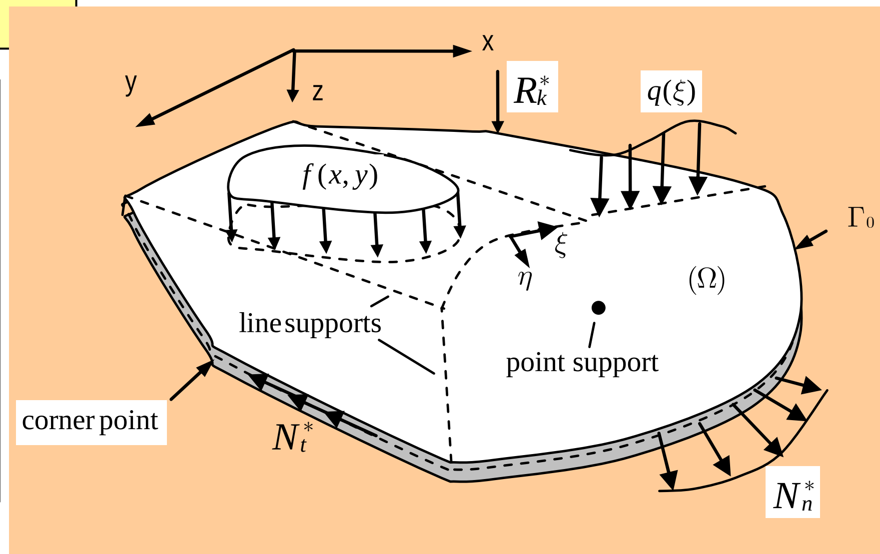
q_x, q_y = inplane body forces

N_x, N_{xy}, N_y = inplane stress resultants

V_n^*, M_n^* = boundary effective shear force and moment

$k_R, k_T^{(k)}$ = rotational and transvese support stiffness

D $= Eh^3/12(1-\nu^2)$ plate stiffness



The boundary operators are defined as

$$V = -D\left[\frac{\partial}{\partial n}\nabla^2 + (1-\nu)\frac{\partial}{\partial s}\left(\frac{\partial^2}{\partial s \partial n} - \kappa\frac{\partial}{\partial s}\right)\right]$$

$$M = -D\left[\nabla^2 - (1-\nu)\left(\frac{\partial^2}{\partial s^2} + \kappa\frac{\partial}{\partial n}\right)\right]$$

$$T = D(1-\nu)\left(\frac{\partial^2}{\partial s \partial n} - \kappa\frac{\partial}{\partial s}\right)$$

$\kappa = \kappa(s)$ curvature of the boundary

The stress resultants

$$Mx = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right), \qquad My = -D\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right)$$

$$Mxy = D(1-\nu)\frac{\partial^2 w}{\partial x \partial y}$$

$$Qx = -D\frac{\partial \nabla^2 w}{\partial x}, \qquad Qy = -D\frac{\partial \nabla^2 w}{\partial y}$$

The AEM solution

The Analog Equation

$\nabla^4 w = b(\mathbf{x})$ $b(\mathbf{x})$ → fictitious source

Using Betti's reciprocal theorem, we establish the Rayleigh-Green identity

$$\int_\Omega (v\nabla^4 w - w\nabla^4 v)\,d\Omega = \int_\Gamma (vVw + w_{,n} Mv - v_{,n} Mw - wVv)\,ds - \sum_k (v[\![Tw]\!] - w[\![Tv]\!])_k$$

Taking

$\nabla^4 v = \delta(\mathbf{x}-\mathbf{y})$, $v = v(\mathbf{x},\mathbf{y}) = \dfrac{1}{8\pi} r^2 \ln r$ is the fundamental solution

We obtain

$$w(\mathbf{x}) = \int_\Omega v b\,d\Omega + \int_\Gamma (vVw + w_{,n} Mv - v_{,n} Mw - wVv)\,ds - \sum_k (v[\![Tw]\!] - w[\![Tv]\!])_k$$

For $x \to \Gamma$ we obtain the boundary integral equations at smooth points

$$\frac{1}{2}w(\mathbf{x}) = \int_\Omega v b \, d\Omega + \int_\Gamma (vVw + w_{,n} Mv - v_{,n} Mw - wVv) ds - \sum_k \left(v [\![Tw]\!] - w [\![Tv]\!] \right)_k \quad \mathbf{x} \in \Gamma$$

$$\frac{1}{2}w_{,\nu}(\mathbf{x}) = \int_\Omega v_1 b \, d\Omega + \int_\Gamma (v_{,\nu} Vw + w_{,n} Mv_{,\nu} - v_{,\nu,n} Mw - wVv_{,\nu}) ds - \sum_k \left(v_{,\nu} [\![Tw]\!] - w [\![Tv_{,\nu}]\!] \right)_k \quad \mathbf{x} \in \Gamma$$

At corner points it is

$$\frac{\alpha}{2\pi}w(\mathbf{x}) = \int_\Omega v b \, d\Omega + \int_\Gamma (vVw + w_{,n} Mv - v_{,n} Mw - wVv) ds - \sum_k \left(v [\![Tw]\!] - w [\![Tv]\!] \right)_k$$

α is the angle between the tangents of the boundary at the corner point;

Derivatives of the solution for points $\mathbf{x} \in \Omega$

$$w_{,x} = \int_{\Omega} v_{,x}\, b\, d\Omega + \int_{\Gamma} (v_{,x} Vw + w_{,n} Mv_{,x} - v_{,nx} Mw - wVv_{,x}) ds - \sum_k \left(v_{,x} [\![Tw]\!] - w [\![Tv_{,x}]\!] \right)_k$$

$$w_{,y} = \int_{\Omega} v_{,y}\, b\, d\Omega + \int_{\Gamma} (v_{,y} Vw + w_{,n} Mv_{,y} - v_{,ny} Mw - wVv_{,y}) ds - \sum_k \left(v_{,y} [\![Tw]\!] - w [\![Tv_{,y}]\!] \right)_k$$

$$w_{,xx} = \int_{\Omega} v_{,xx}\, b\, d\Omega + \int_{\Gamma} (v_{,xx} Vw + w_{,n} Mv_{,xx} - v_{,nxx} Mw - wVv_{,xx}) ds - \sum_k \left(v_{,xx} [\![Tw]\!] - w [\![Tv_{,xx}]\!] \right)_k$$

$$w_{,xy} = \int_{\Omega} v_{,xy}\, b\, d\Omega + \int_{\Gamma} (v_{,xy} Vw + w_{,n} Mv_{,xy} - v_{,nxy} Mw - wVv_{,xy}) ds - \sum_k \left(v_{,xy} [\![Tw]\!] - w [\![Tv_{,xy}]\!] \right)_k$$

$$w_{,yy} = \int_{\Omega} v_{,yy}\, b\, d\Omega + \int_{\Gamma} (v_{,yy} Vw + w_{,n} Mv_{,yy} - v_{,nyy} Mw - wVv_{,yy}) ds - \sum_k \left(v_{,yy} [\![Tw]\!] - w [\![Tv_{,yy}]\!] \right)_k$$

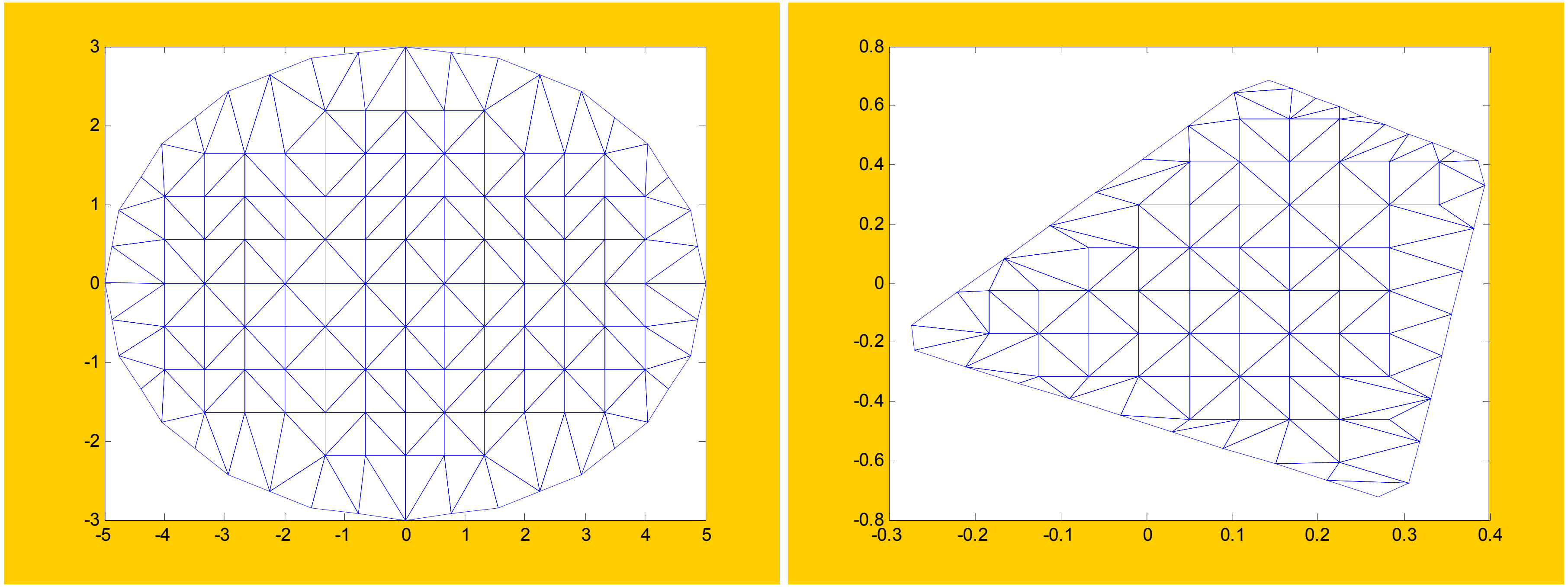
Evaluation of the domain integral $\int_\Omega v b \, d\Omega$

(a) Domain discretization

The domain Ω is divided into N cells and $b(\mathbf{x})$ is approximated as in FEM.

Triangular cells can be automatically established from a set of nodal points using Delaunay Triangulation

$$\int_\Omega v b \, d\Omega = \sum_{j=1}^{M} F_{xj} b^j \qquad F_{xj} = \int_j v(\mathbf{x}, \mathbf{y}) \, d\Omega_j \quad j = 1, 2 \ldots, M,$$



(b) Conversion of the domain integral to boundary line integral

$$b = \sum_{j=1}^{M} a_j f_j \qquad f_j = \sqrt{c^2 + r^2} \quad \text{radial basis functions}$$

$$\int_\Omega v\, \boldsymbol{b} d\Omega = \sum_{k=1}^{M} a_k \int_\Omega v f_k d\Omega$$

Using the Raleigh-Green identity

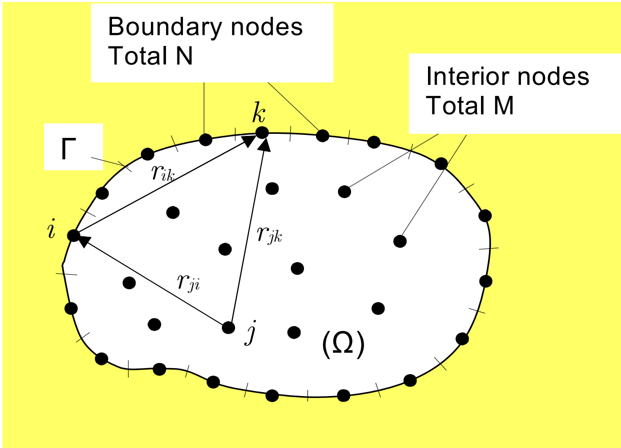
$$\int_\Omega v f_k(r) d\Omega = \hat{w} + \int_\Gamma \left(v \frac{\partial}{\partial n}\nabla^2 \hat{w} - \hat{w}\frac{\partial}{\partial n}\nabla^2 v - v_{,n}\nabla^2 \hat{w} - \hat{w}_{,n}\nabla^2 v\right) d\Omega$$

$\hat{w}_k$ is a particular solution of the equation

$$\nabla^4 \hat{w}_k = f_k$$

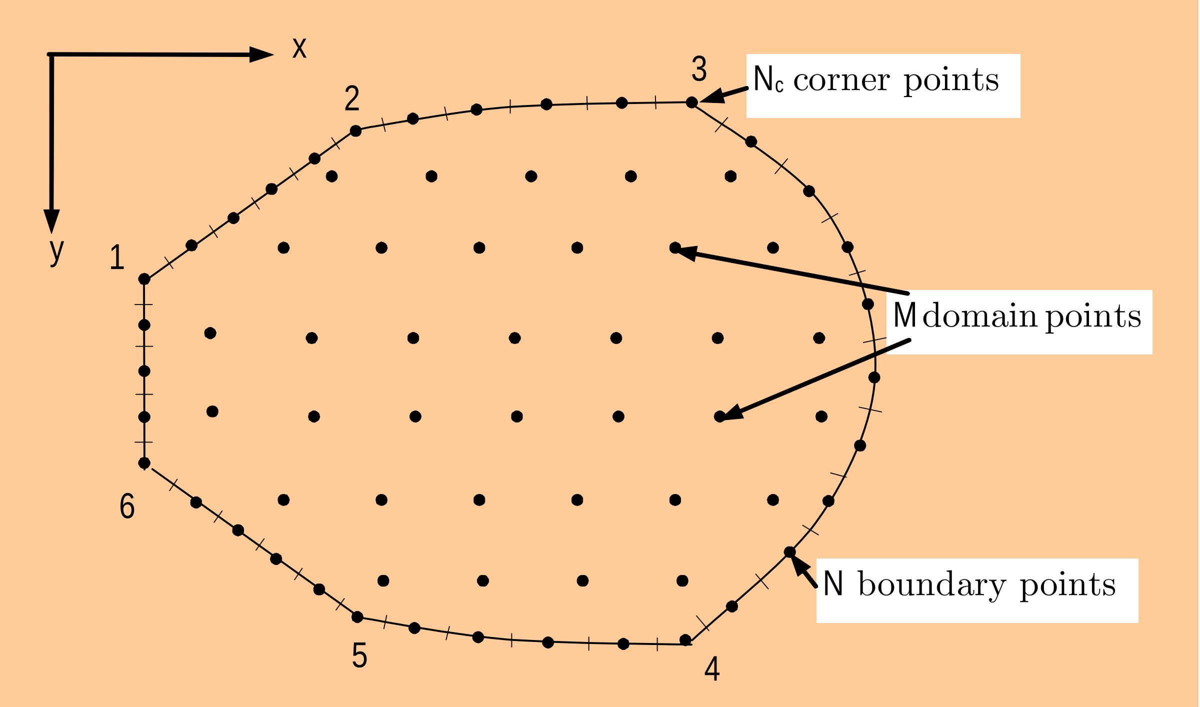
for $f_k = \sqrt{r^2 + c_j^2}, \quad r = \| x - x_k \|,$ c=shape parameter

$$\hat{w}_k = \left(\frac{1}{30}c^5 - \frac{1}{12}r^2c^3\right)\ln(f + c) - \frac{61}{900}c^4 f + \frac{1}{225}r^4 f + \frac{4}{75}r^2c^2 f$$



Numerical implementation

BEM with N constant Boundary Elements and M domain meshless collocation points



Domain and Boundary discretization

The Boundary Integral Equations and the BCs are applied at the N+Nc boundary nodal points

The Differential Equation is applied at the M domain nodal points

$$\mathbf{H}\begin{Bmatrix}\mathbf{w}\\ \mathbf{w}_{,c}\\ \mathbf{w}_{,n}\end{Bmatrix}=\mathbf{G}\begin{Bmatrix}\mathbf{V}\\ \mathbf{R}\\ \mathbf{M}\end{Bmatrix}+\mathbf{Aa} \qquad (2N+N_c \text{ equations})$$

$$\alpha_1\mathbf{w}+\alpha_2\mathbf{V}=\alpha_3$$
$$\beta_1\mathbf{w}_{,n}+\beta_2\mathbf{M}=\beta_3 \qquad (2N+N_c \text{ equations})$$
$$\mathbf{c}_1\mathbf{w}_c+\mathbf{c}_2\mathbf{R}=\mathbf{c}_3$$

$$\mathbf{Db}-(\mathbf{N}_x\mathbf{w}_{,xx}+2\mathbf{N}_{xy}\mathbf{w}_{,xy}+\mathbf{N}_y W_{,yy})+\mathbf{q}_x\mathbf{w}_{,x}+\mathbf{q}_y\mathbf{w}_{,y}=\mathbf{f}$$

or

$$\mathbf{Aa=c} \qquad (M \text{ equations})$$

Total unknowns	$4N+2Nc+M$
Available equations	$4N+2Nc+M$

$\mathbf{H}$, $\mathbf{G}$, $\mathbf{A}$:	known coefficient matrices
$\mathbf{w}$, $\mathbf{w}_c$, $\mathbf{w}_{,n}$:	the vectors of the boundary nodal displacements, corner displacements and boundary nodal slopes, respectively. (geometrical quantities)
$\mathbf{V}$, $\mathbf{R}$, $\mathbf{M}$:	the vectors of boundary nodal effective shear forces, the concentrated corner forces and boundary nodal normal bending moments (physical quntities)
$\mathbf{a}$:	is the vector of the unknown coefficients
$\alpha_1,\alpha_2,\ldots,\mathbf{c}_2$	known coefficient matrices
$\alpha_3,\beta_3,\ldots,\mathbf{c}_3$	known vectors.

Evaluation of the solution and the stress resultants

After evaluating the boundary quantities $\mathbf{w}, \mathbf{w}_{,c}, \mathbf{w}_{,n}, \mathbf{V}, \mathbf{R}, \mathbf{M}$ and the coefficients $\mathbf{a}$, the deflection and the its derivatives at a point $\mathbf{x} \in \Omega$, hence the stress resultants are obtained from their discteretized counterparts of the respective integral representation, namely

1. Deflections

$$w(\mathbf{x}) = \tilde{\mathbf{H}}(\mathbf{x}) \begin{Bmatrix} \mathbf{w} \\ \mathbf{w}_{,c} \\ \mathbf{w}_{,n} \end{Bmatrix} - \mathbf{G}(\mathbf{x}) \begin{Bmatrix} \mathbf{V} \\ \mathbf{R} \\ \mathbf{M} \end{Bmatrix} + \mathbf{A}(\mathbf{x})\mathbf{a}$$

2. Derivatives

$$w_{,pqr}(\mathbf{x}) = \tilde{\mathbf{H}}_{,pqr}(\mathbf{x}) \begin{Bmatrix} \mathbf{w} \\ \mathbf{w}_{,c} \\ \mathbf{w}_{,n} \end{Bmatrix} - \mathbf{G}_{,pqr}(\mathbf{x}) \begin{Bmatrix} \mathbf{V} \\ \mathbf{R} \\ \mathbf{M} \end{Bmatrix} + \mathbf{A}_{,pqr}(\mathbf{x})\mathbf{a}, \qquad p,q,r = 0,x,y$$

3. Stress resultants

$$Mx = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right), \qquad My = -D\left(\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2}\right) \qquad Mxy = D(1-\nu)\frac{\partial^2 w}{\partial x \partial y}$$

$$Qx = -D\frac{\partial \nabla^2 w}{\partial x}, \qquad Qy = -D\frac{\partial \nabla^2 w}{\partial y}$$

Variational Solution [13]

The solution after elimination of the boundary quantities can be also put in the form

$$w(\mathbf{x}) = \sum_{k=1}^{M} a_k W_k(\mathbf{x}) + W_0(\mathbf{x}) \qquad \mathbf{x} \in \Omega \qquad \text{Ritz expansion } a_k := \text{Ritz coefficients}$$

$$\left.\begin{array}{ll} W_k(\mathbf{x}) & \text{satisfies homogeneous BCs} \\ W_0(\mathbf{x}) & \text{satisfies inhomogeneous BCs both kinematic and natural} \end{array}\right\} \text{admissible shape functions}$$

Similarly, the derivatives

$$w_{,pqr}(\mathbf{x}) = \sum_{k=1}^{M} a_k W_{k,pqr}(\mathbf{x}) + W_{0,pqr}(\mathbf{x}) \qquad \mathbf{x} \in \Omega \qquad p,q,r = 0,x,y$$

Total Potential = Functional to be minimized

$$\Pi(w) = \frac{D}{2}\int_{\Omega}\left[w_{,xx}^2 + w_{,yy}^2 + 2\nu w_{,xx} w_{,yy} + 2(1-\nu) w_{,xy}^2\right] d\Omega$$
$$+\frac{1}{2}\int_{\Omega}(N_x w_{,x}^2 + N_y w_{,y}^2 + 2N_{xy} w_{,x} w_{,y})\, d\Omega + \frac{1}{2}\int_{\Gamma}\left(k_T w^2 + k_R w_{,n}^2\right) ds$$
$$-\int_{\Omega} f w\, d\Omega - \int_{\Gamma}\left(V_n^* w - M_n^* w_{,n}\right) ds - \sum_k R_k w^{(k)}$$

Π depends on the following sets of parameters:

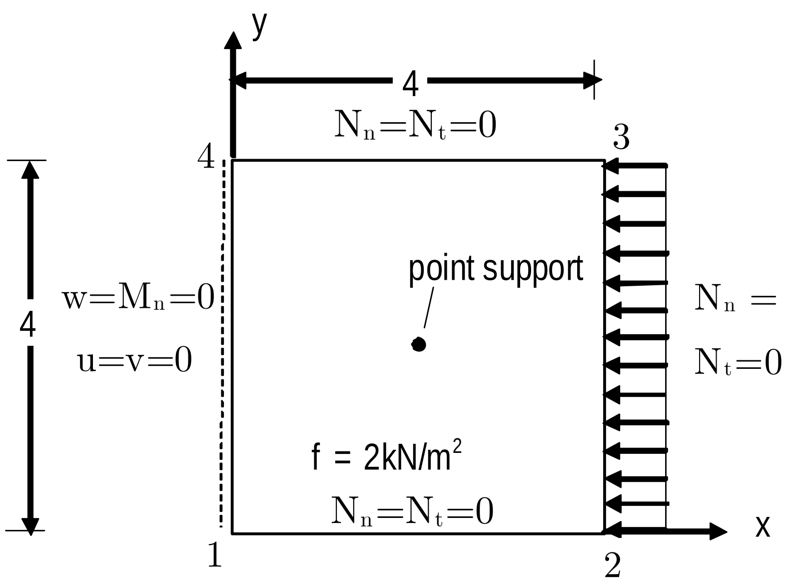
a) The Ritz coefficients $a_j, a_j^{(1)}, a_j^{(2)}$, j=1,2,...,M

b) The shape parameters c_j of the multiquadrics-RBF

c) The 2M coordinates of the collocation points

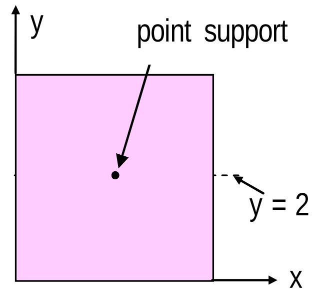
4. Numerical examples

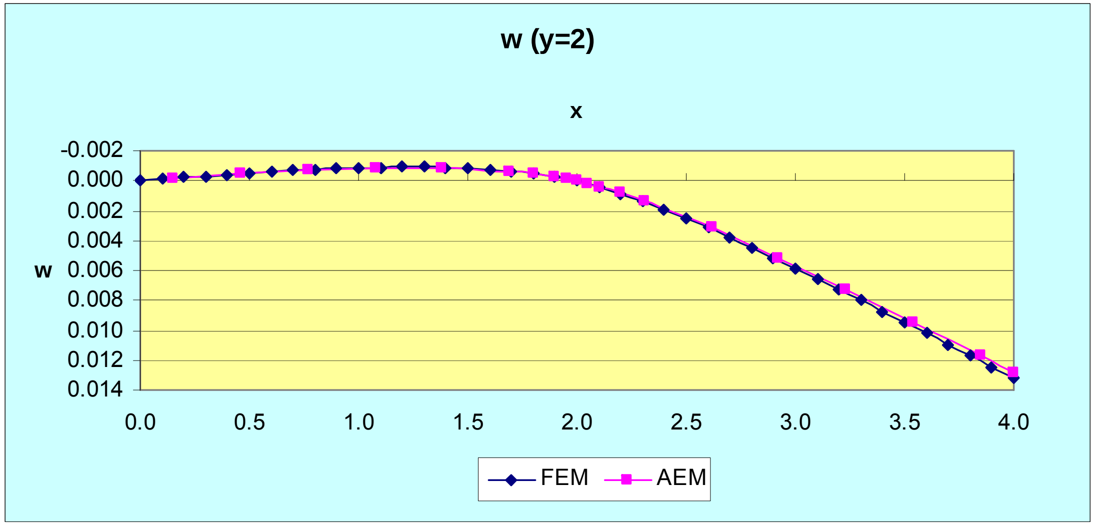
Example 1 : Rectangular plate under transverse and membrane forces

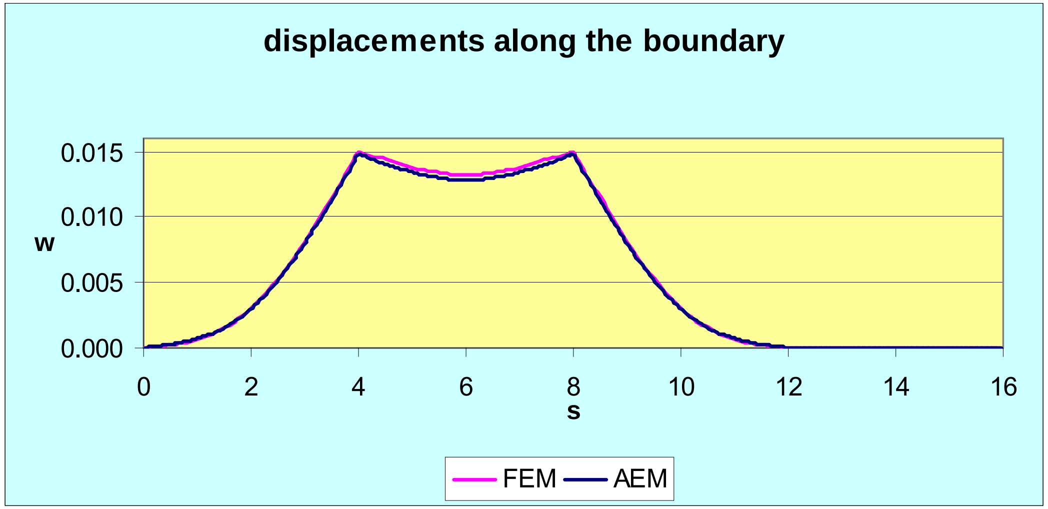


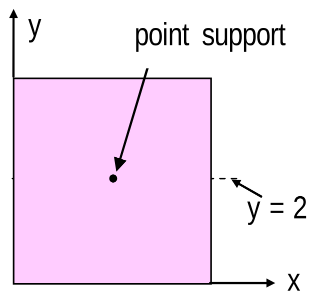
Data : $h = 0.08\text{m}$, $E = 30 \times 10^6\ \text{kN/m}^2$, $\nu = 0.2$, $f = 2\,\text{kN/m}^2$ $N = 320$ $M = 169$ $c_{optimal} = 0.1$

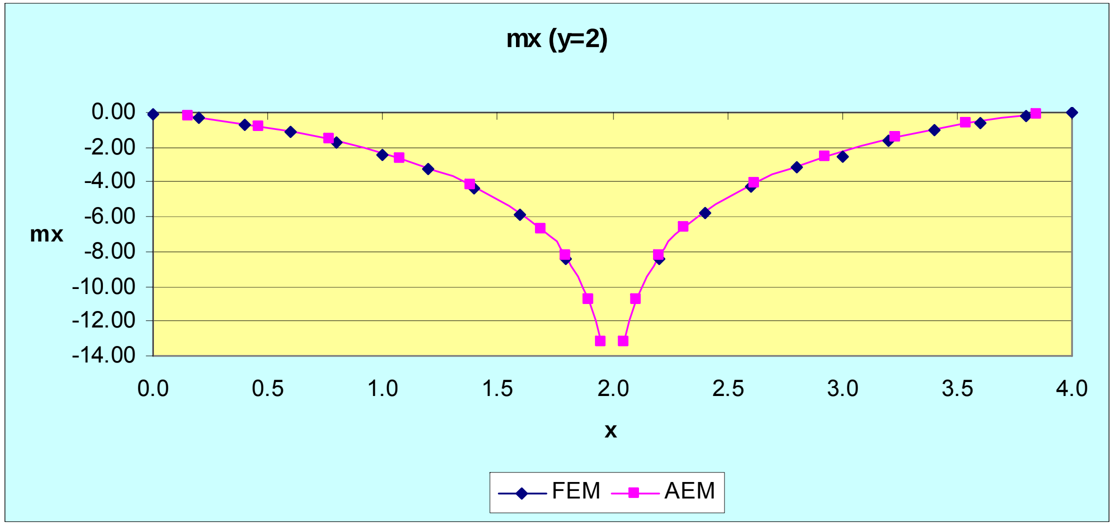
Point support P=37.63 kN (FEM P=37.55kN)

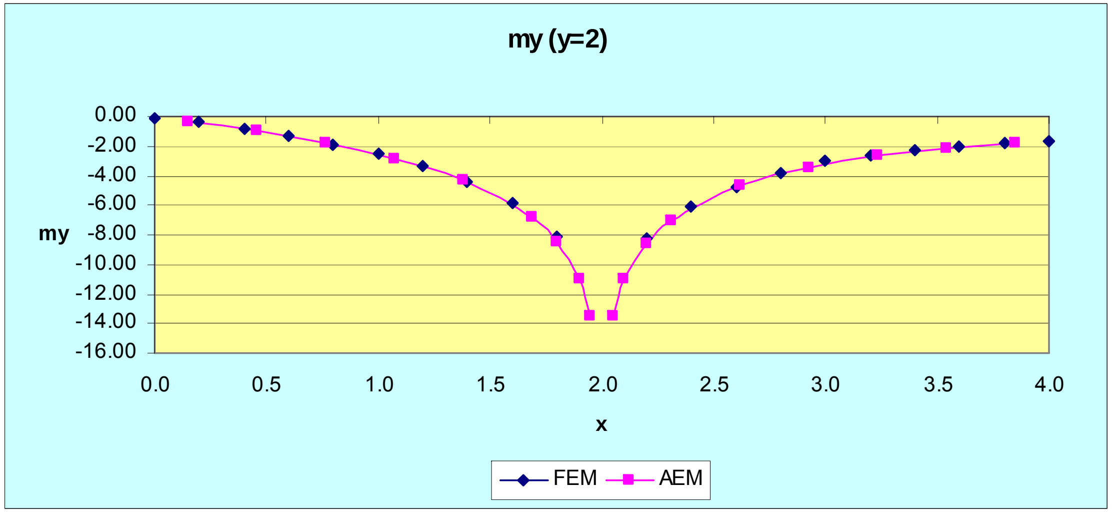

y
point support
y = 2
x


w (y=2)
x
-0.002
0.000
0.002
0.004
0.006
0.008
0.010
0.012
0.014
w
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
FEM
AEM

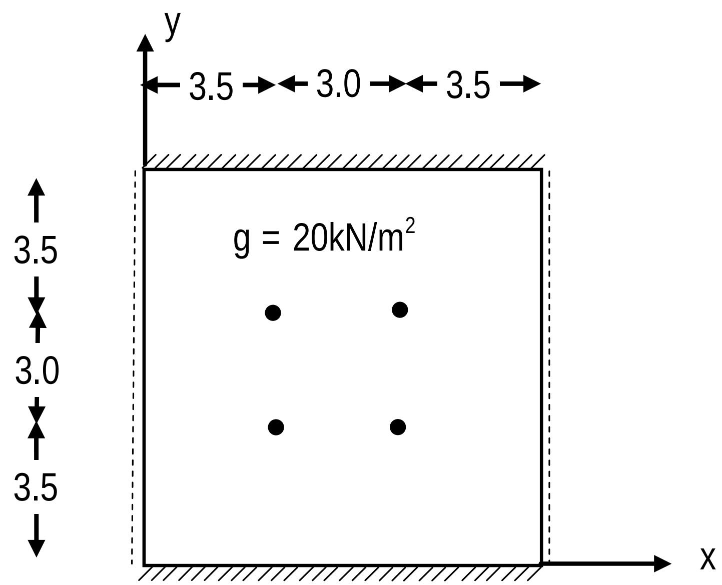

displacements along the boundary
0.015
0.010
0.005
0.000
w
0
2
4
6
8
10
12
14
16
s
FEM
AEM


y
point support
y = 2
x


mx (y=2)
mx
0.00
-2.00
-4.00
-6.00
-8.00
-10.00
-12.00
-14.00
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
x
FEM
AEM


my (y=2)
my
0.00
-2.00
-4.00
-6.00
-8.00
-10.00
-12.00
-14.00
-16.00
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
x
FEM
AEM

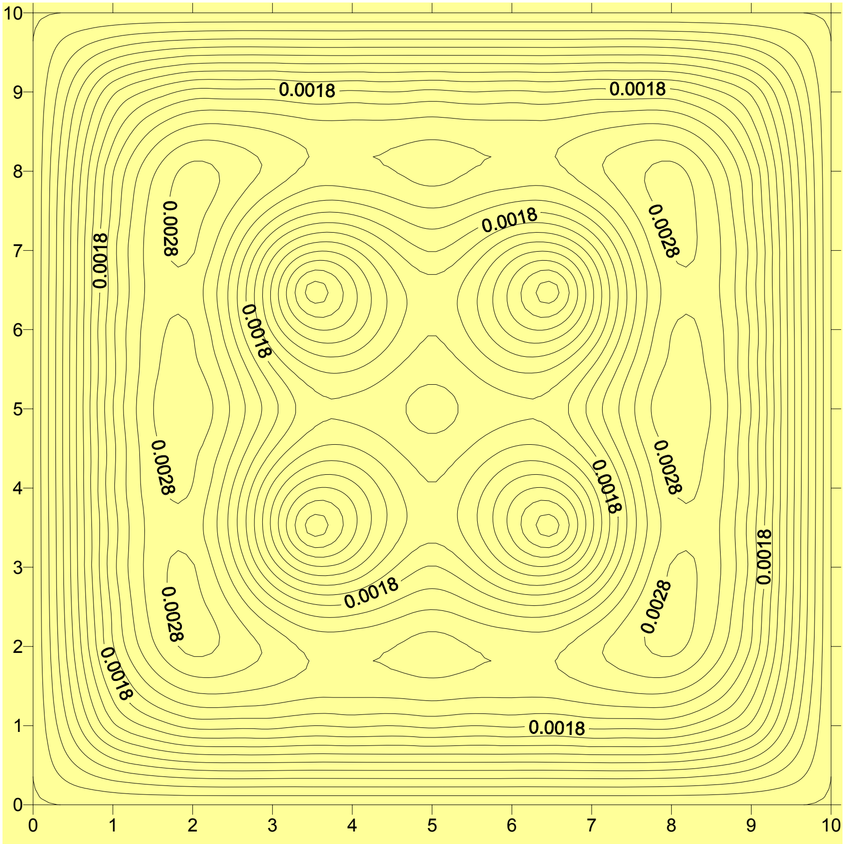
Example 2. Square plate on biparametric elastic foundation in addition to four piles



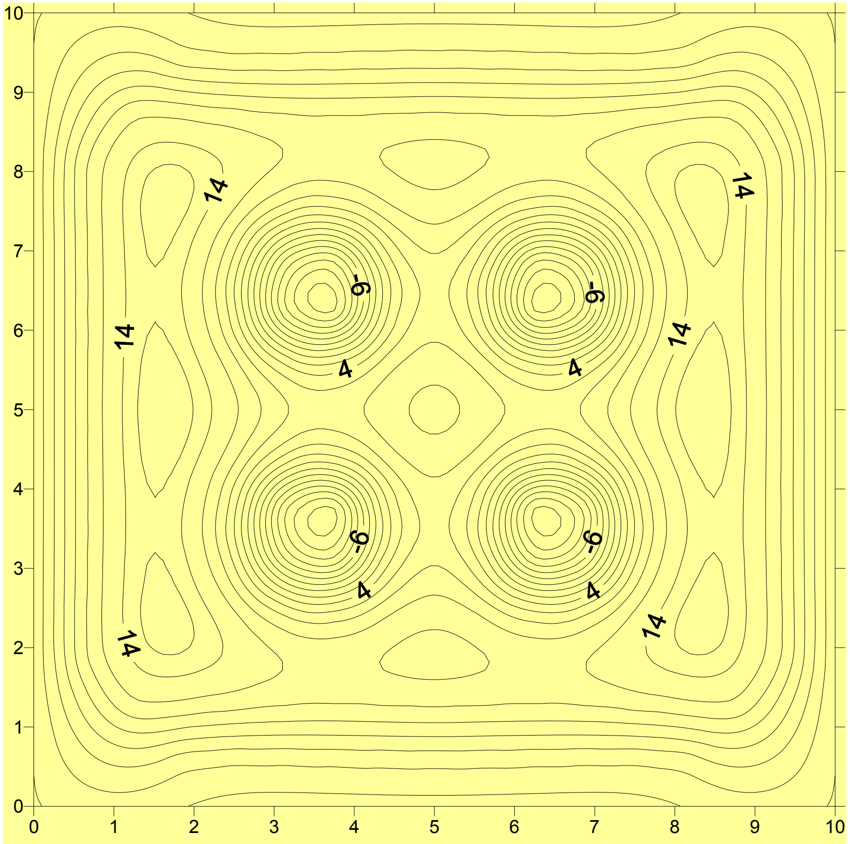
Data : $h = 0.1\text{m},\ E{=}30\times10^6\ \text{kN/m}^2,\ \nu = 0.3,\ f = 20\,\text{kN/m}^2$ $N = 200$ $M = 81$ $c_{optimal} = 1.05$

subgrade : $p = kw - G\nabla^2 w,\ \ k{=}G{=}2747.5\ \text{kN/m}^2$

Point support reaction P=146.5 kN (Ref.[9] P=147.5 kN)



Deflection w



Subgrade reaction

Non linear 4th order PDEs. Large Deflections of plates

Transverse deflections

$$D\nabla^4 w - (N_x w_{,xx} + 2N_{xy} w_{,xy} + N_y w_{,yy}) + b_x w_{,x} + b_y w_{,y} + p_s = f \quad \text{in } \Omega$$

$$Vw + N_n^* w_{,n} + N_t^* w_{,t} + k_T w = V_n^* \quad \text{or } w = w^* \quad \text{on } \Gamma$$

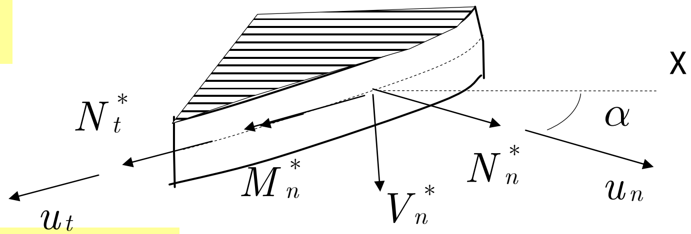
$$Mw + k_R w_{,n} = M_n^* \quad \text{or } w_{,n} = w_{,n}^* \quad \text{on } \Gamma$$

$$k_T^{(k)} w^{(k)} - [\![Tw]\!]_k = R_k^* \quad \text{or } w^{(k)} = w_k^* \quad \text{at corner point k}$$

N_x, N_y, N_{xy} : memrbane forces

$V_n^*, M_n^*, N_n^*, N_t^*$: boundary forces

R_k^* : corner forces



Inplane deformation. (Plane stress problem)

$$\nabla^2 u + \frac{1+\nu}{1-\nu}(u_{,x} + v_{,y})_{,x} + w_{,x}\left(\frac{2}{1-\nu} w_{,xx} + w_{,yy}\right) + \frac{1+\nu}{1-\nu} w_{,x} w_{,y} + \frac{b_x}{G} = 0 \quad \text{in } \Omega$$

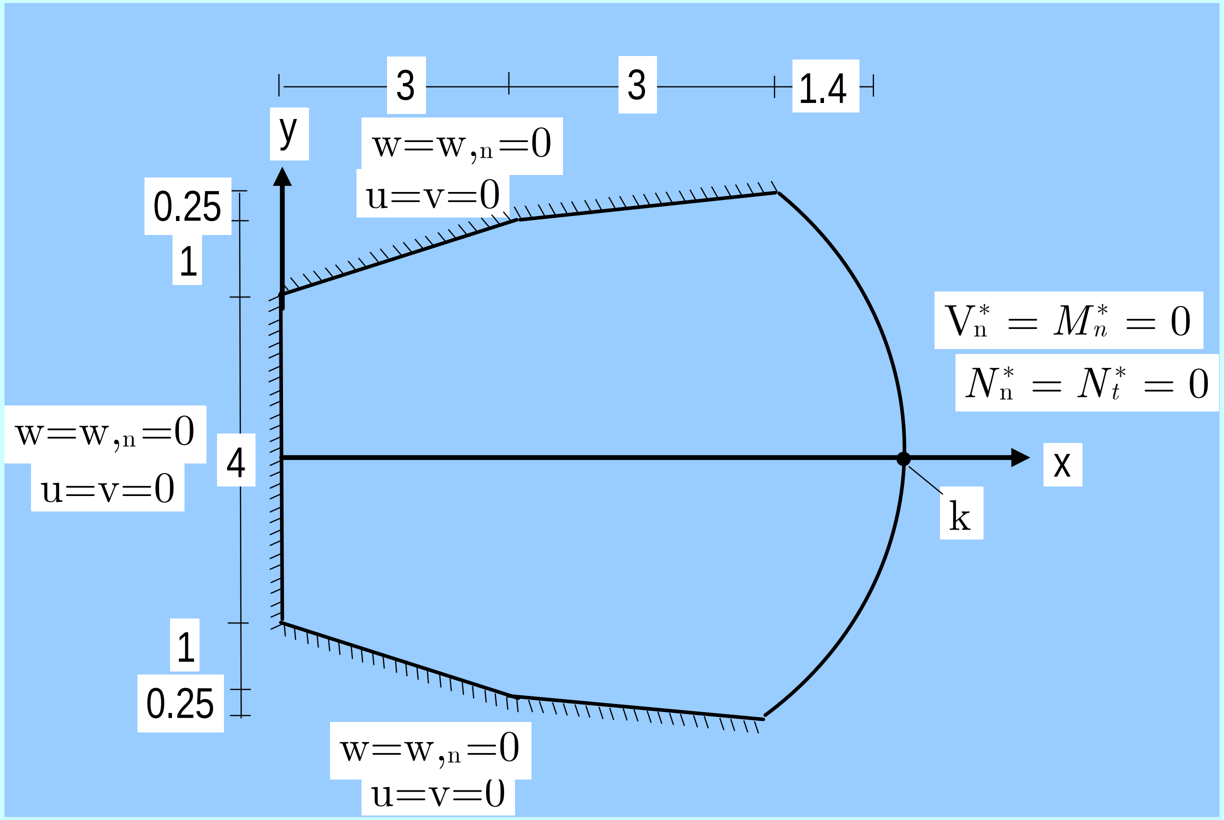
$$\nabla^2 v + \frac{1+\nu}{1-\nu}(u_{,x} + v_{,y})_{,y} + w_{,y}\left(\frac{2}{1-\nu} w_{,yy} + w_{,xx}\right) + \frac{1+\nu}{1-\nu} w_{,x} w_{,y} + \frac{b_y}{G} = 0 \quad \text{in } \Omega$$

For linear problems these terms vanish

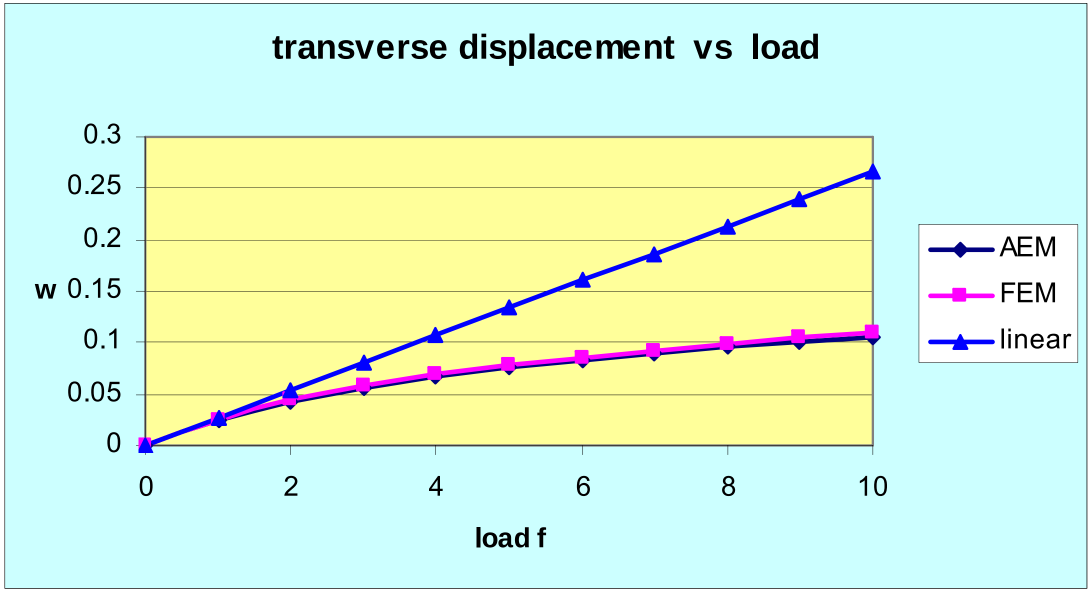
$$N_n = N_n^* \quad \text{or} \quad u_n = u_n^* \quad \text{on } \Gamma$$

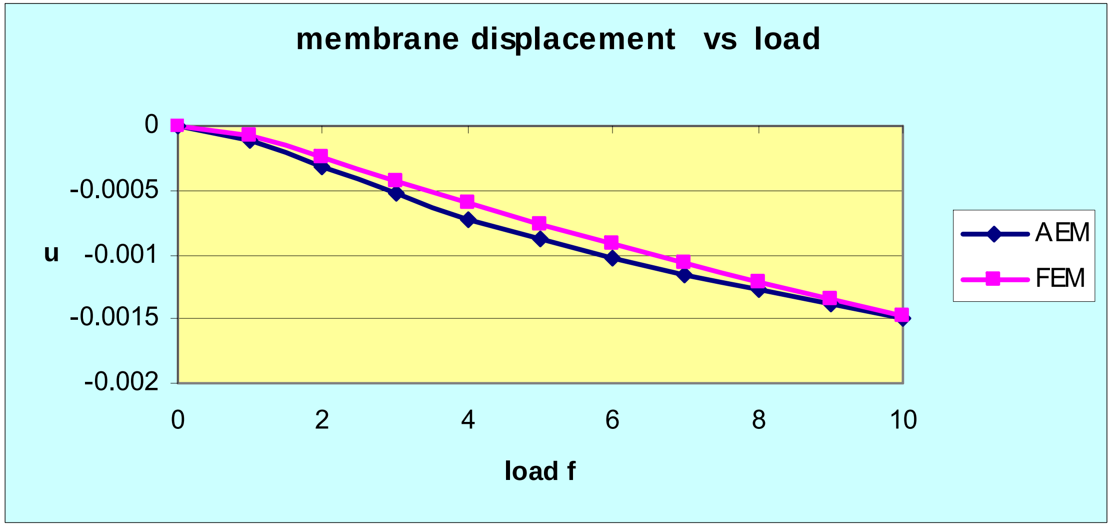
$$N_t = N_t^* \quad \text{or} \quad u_t = u_t^* \quad \text{on } \Gamma$$

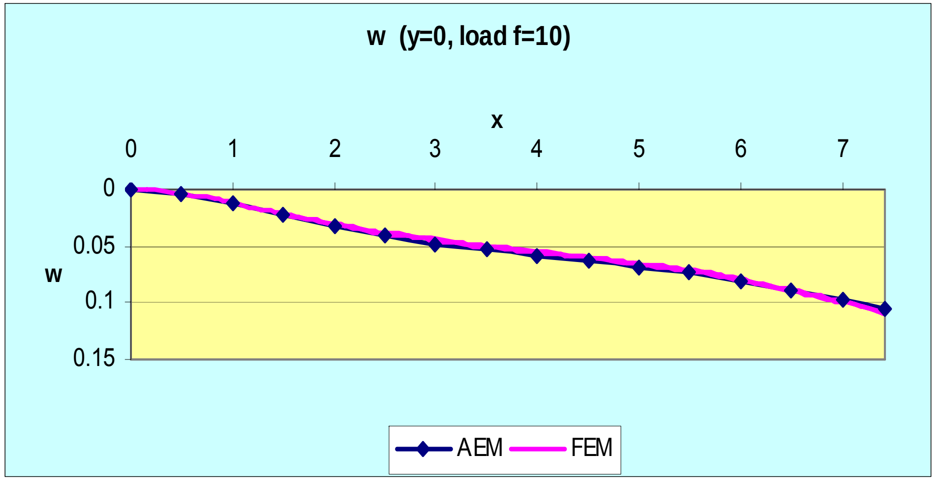
Example 4. Large deflections of a plate with complex geometry

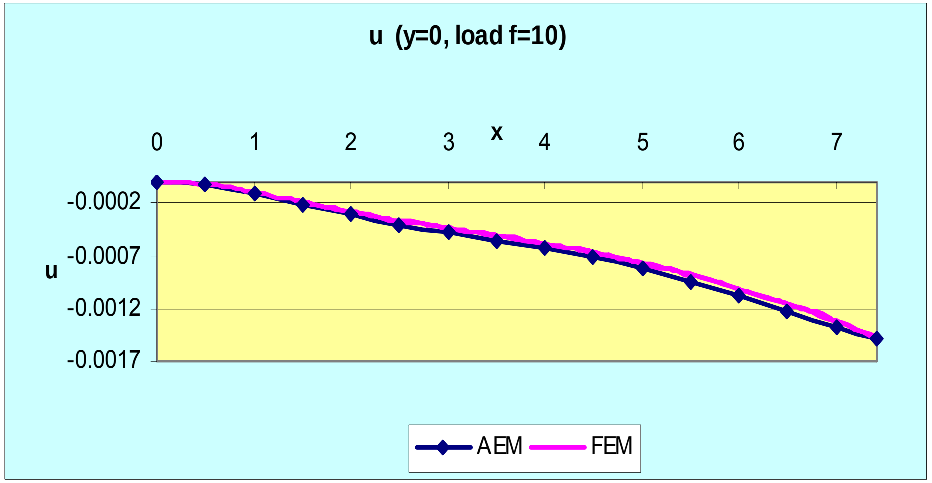


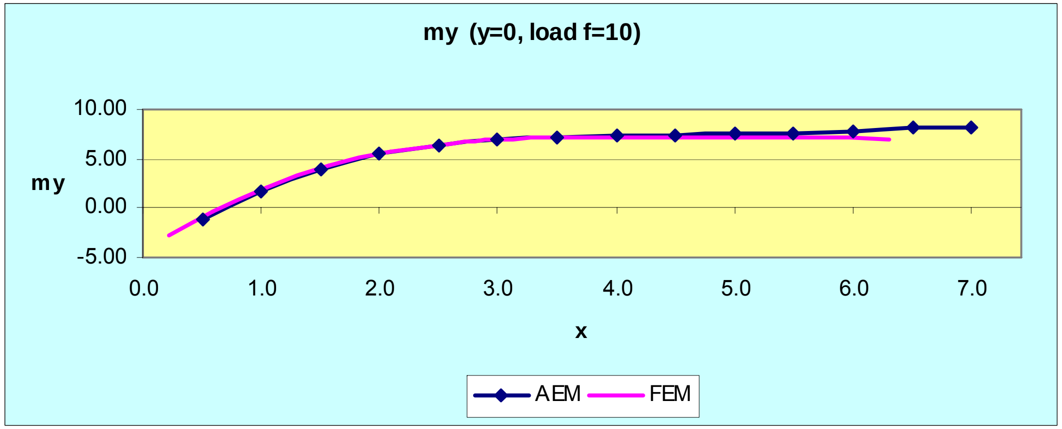
Data : $h = 0.05\,m$, $E = 30\times 10^6\ kN/m^2$, $\nu = 0.3$, $N = 368$ $M = 136$ $c_{optimal} = 0.01$


transverse displacement vs load
w
0.3
0.25
0.2
0.15
0.1
0.05
0
0
2
4
6
8
10
load f
AEM
FEM
linear


membrane displacement vs load
u
0
-0.0005
-0.001
-0.0015
-0.002
0
2
4
6
8
10
load f
AEM
FEM

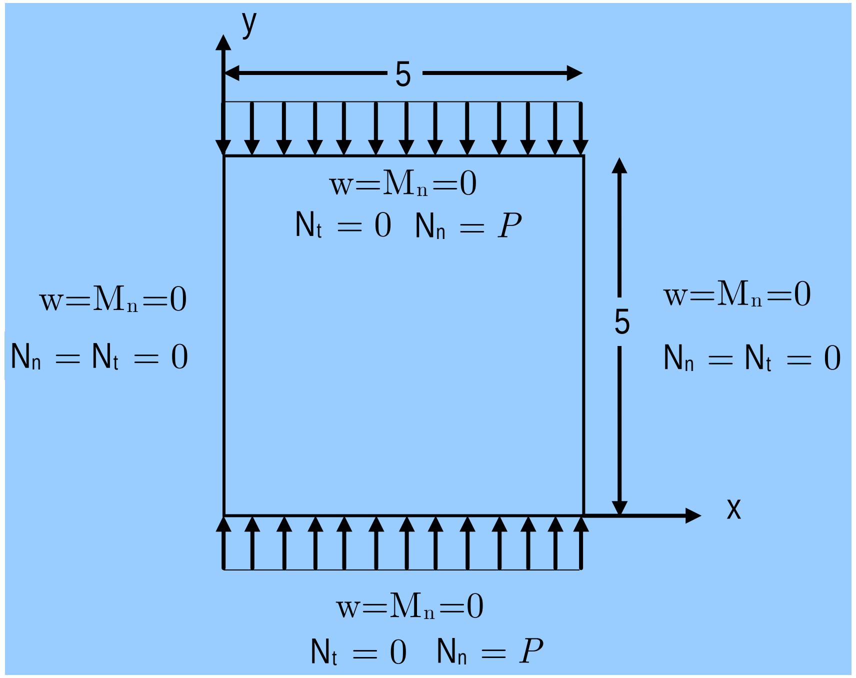

w (y=0, load f=10)
x
0
1
2
3
4
5
6
7
0
0.05
w
0.1
0.15
AEM
FEM


u (y=0, load f=10)
0
1
2
3
x
4
5
6
7
-0.0002
u
-0.0007
-0.0012
-0.0017
AEM
FEM


my (y=0, load f=10)
10.00
5.00
my
0.00
-5.00
0.0
1.0
2.0
3.0
4.0
5.0
6.0
7.0
x
AEM
FEM

Post buckling analysis of plates [24]

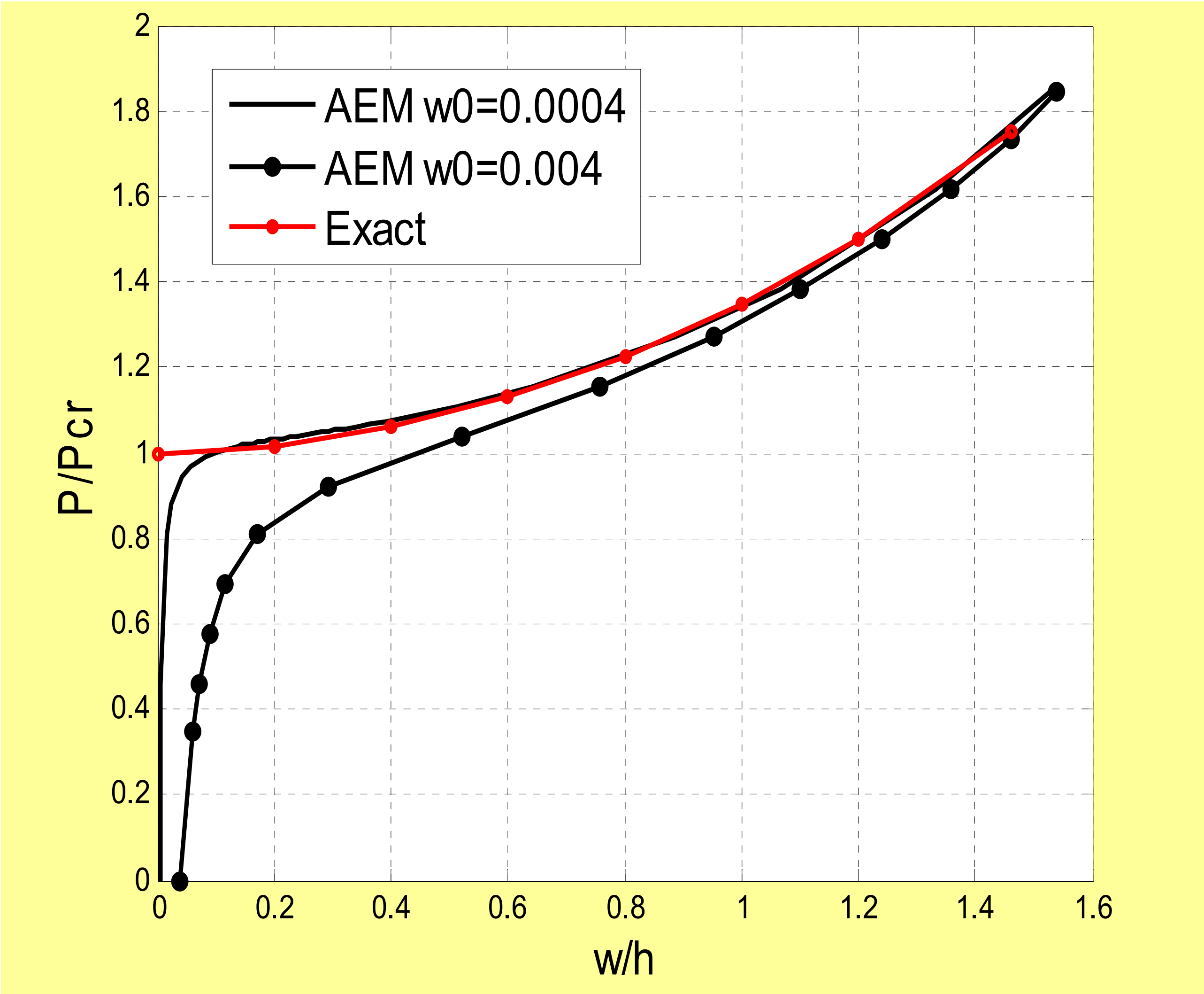
Example 1 : Square plate with simply supported movable edges



Data : $h = 0.1m$, $E = 30 \times 10^6$ kN/m^2, $\nu = 0.3$, $N = 200$ $M = 81$ $c_{optimal} = 0.6$

$P_{cr}^1 = 4332.3kN$ (exact 4333.3kN)

Load – deflection curve at the center of the plate



Load – deflection curves for different initial deflection forms