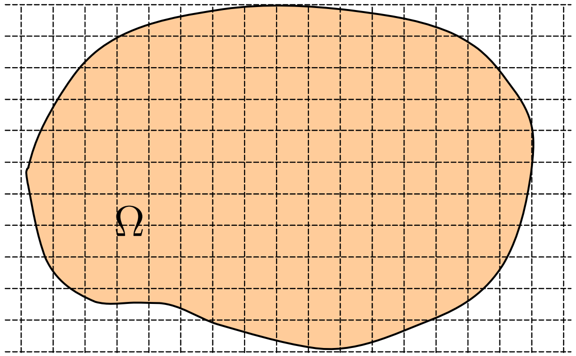
Part A

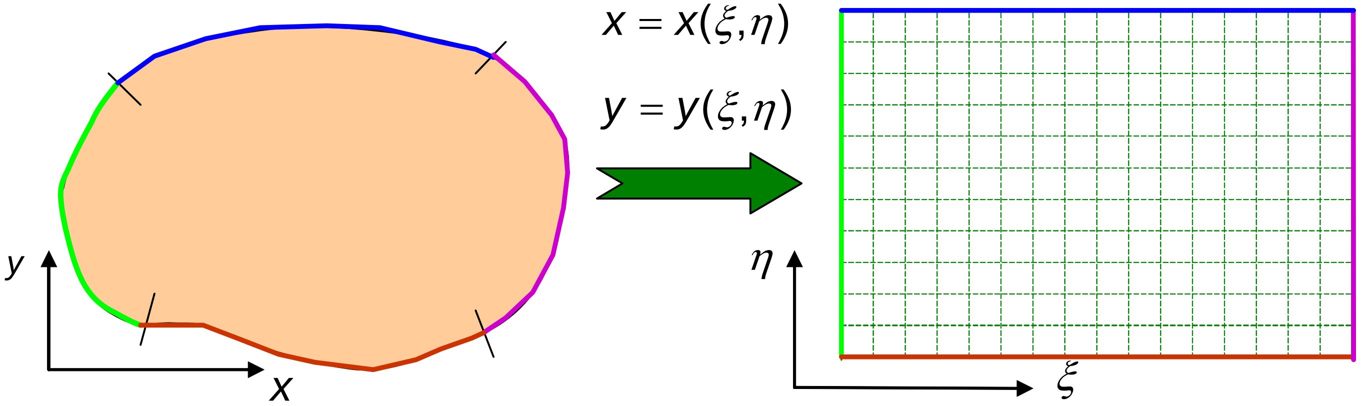
The Conventional BEM

The most effective numerical methods for solving PDEs

1. Finite Difference Method



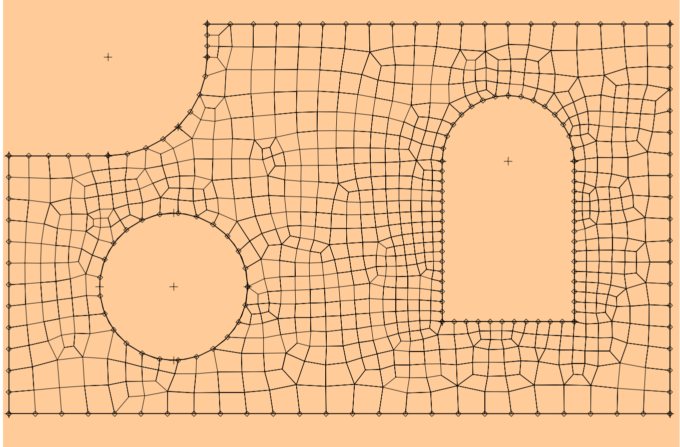
- Gives accurate results.
- Not suitable for arbitrary domains



$$x = x(\xi,\eta)$$

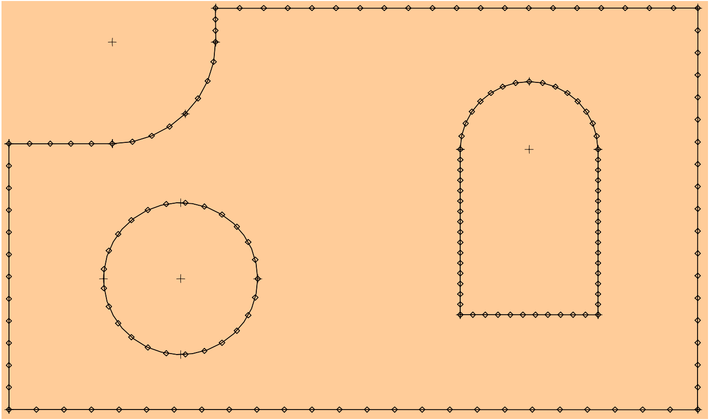
$$y = y(\xi,\eta)$$

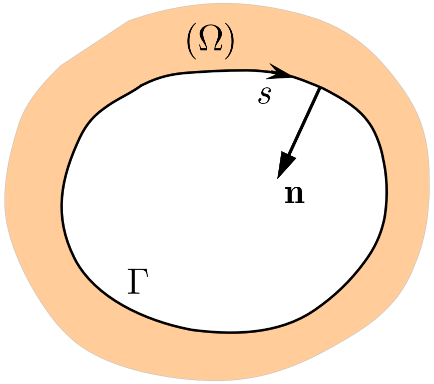
Finite Element Method



- Most popular numerical method for engineering analysis.
- Good for linear and nonlinear problems
- Requires domain discretization, which is a very difficult task for complicated domains.
- Large number of unknowns is required to get accurate results especially for derivatives.
- Ineffective for infinite domains

The Boundary Element Method [1]





(b) Open domain

- Powerful computational method.
- Requires only boundary discretization. Suitable for complicated domains.
- Small number of unknowns to get accurate results.
- Great accuracy of the derivatives
- Suitable for infinite domains
- Reduces the dimension by one
- Pure BEM is good only for linear problems

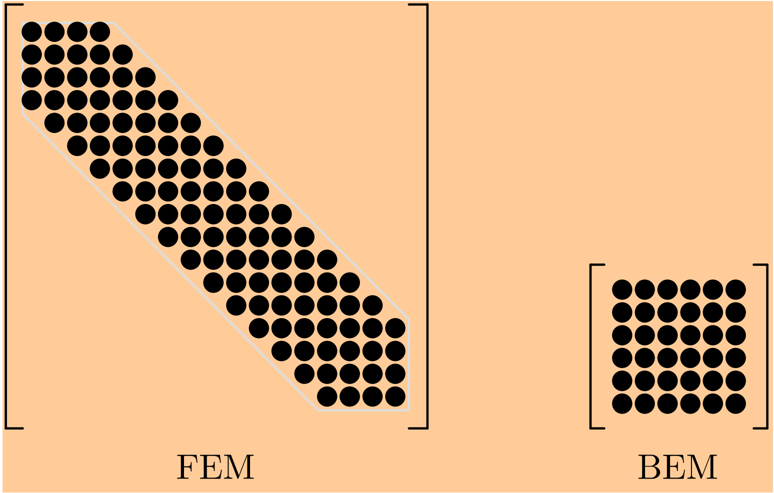
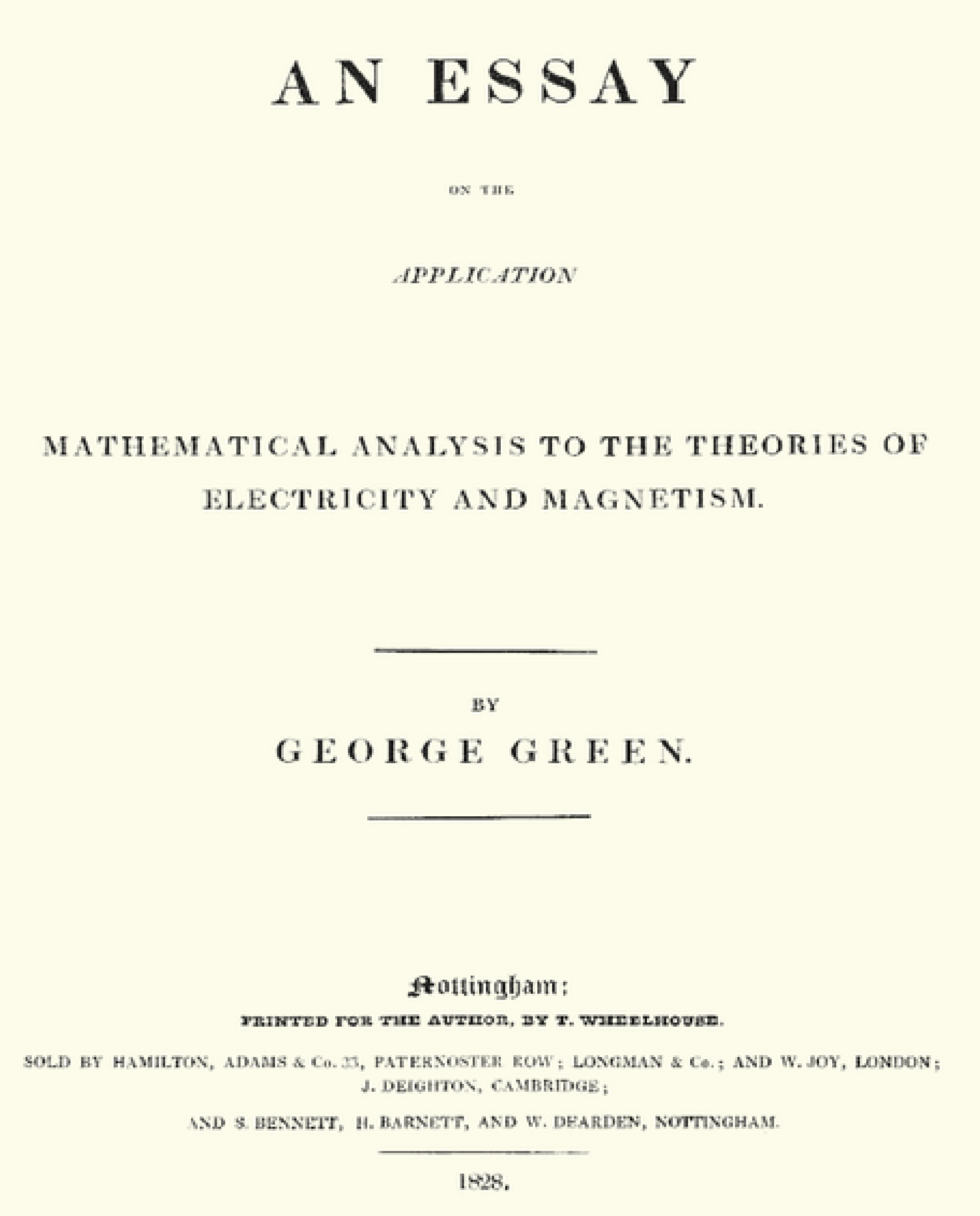


Figure 1.2 Coefficient matrices for FEM and BEM.

The Origin of Boundary Integral Equation Methods

1. The method of singularities of G. Green (1828) for the Laplace equation
2. Fredholm’s method (1903).
3. The BEM (Decade of 1960)

1. The method of singularities of G. Green (1828) for the Laplace equation



AN ESSAY

ON THE

APPLICATION

OF

MATHEMATICAL ANALYSIS TO THE THEORIES OF
ELECTRICITY AND MAGNETISM.

BY

GEORGE GREEN.

Nottingham:
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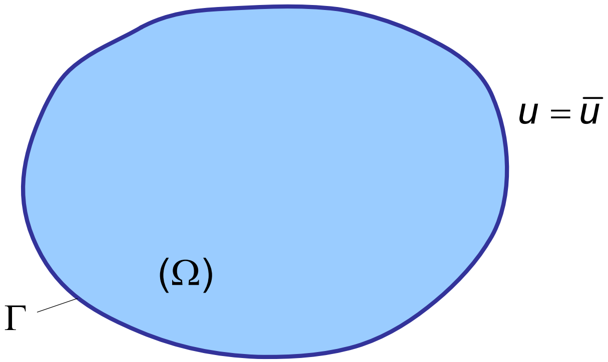
1828.

1.1 *Dirichlet* problem

$\nabla^2 u = 0$ in Ω (1a)

$u = \bar{u}$ on Γ (1b)



Domain with Dirichlet BCs

Green's adjoint problem

$\nabla^2 G = \delta(Q-P)$ in Ω (2a)

$G = 0$ on Γ (2b)

$G(Q,P)$ is the Green's function for the Dirichlet Problem

$\delta(Q-P)$: The Dirac delta function two dimensions is defined as

$$\int_\Omega \delta(Q-P)h(Q)\,d\Omega_Q = h(P), \qquad Q(\xi,\eta), P(x,y) \in \Omega \tag{3}$$

Green's reciprocal identity

$$\int_\Omega \left(v\nabla^2 u - u\nabla^2 v\right) d\Omega = \int_\Gamma \left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\right) ds \tag{4}$$

$$
\begin{aligned}
&\nabla^2 u = 0 \quad \text{in} \ \Omega \\
&u = \bar{u} \quad \text{on} \quad \Gamma \\
\\
&\nabla^2 G = \delta(Q-P) \quad \text{in} \ \Omega \\
&G = 0 \quad \text{on} \quad \Gamma
\end{aligned}
$$

Choose u satisfying BVP (1a,b)

Choose $v = G(Q,P)$ satisfying BVP (2b)

and apply (4)

$$\int_\Omega \left[G\nabla^2 u - u\delta(Q-P) \right] d\Omega = \int_\Gamma \left(G\frac{\partial u}{\partial n} - u\frac{\partial G}{\partial n} \right) ds \tag{5}$$

Solution

$$u(P) = \int_\Gamma \left[\bar{u}(q)\frac{\partial G(q,P)}{\partial n_q} \right] ds_q \tag{5a}$$

or after dropping the arguments

$$u = \int_\Gamma \bar{u} G_{,n}\, ds \tag{5b}$$

A1.2 Neumann problem

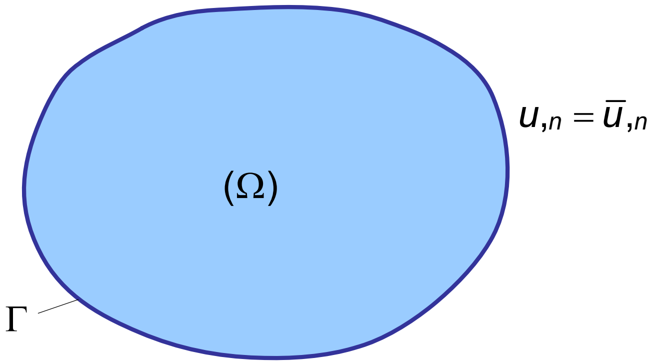
$\nabla^2 u = 0$ in Ω (6a)

$u_{,n} = \bar{u}_{,n}$ on Γ (6b)

Green's adjoint problem

$\nabla^2 G = \delta(Q-P)$ in Ω (7a)

$G_{,n} = 0$ on Γ (7b)



Domain with Neumann BC's

$G(Q,P)$ is the Green's function for the Dirichlet Problem

Choose u satisfying BVP (6a,b)

Choose $v = G(Q,P)$ satisfying BVP (6b)

and apply Green's identity (4)

$$\int_\Omega \left[G\nabla^2 u - u\delta(Q-P) \right] d\Omega = \int_\Gamma \left(G\frac{\partial u}{\partial n} - u\frac{\partial G}{\partial n} \right) ds \qquad (8)$$

Solution

$$u(P) = -\int_\Gamma \left[G(P,q)\bar{u}_{,n}(q) \right] ds_q \qquad (9a)$$

$$u = \int_\Gamma G\bar{u}_{,n}\, ds \qquad (9b)$$

1.3 Mixed BC's

$$\nabla^2 u = 0 \quad \text{in} \quad \Omega \tag{10a}$$

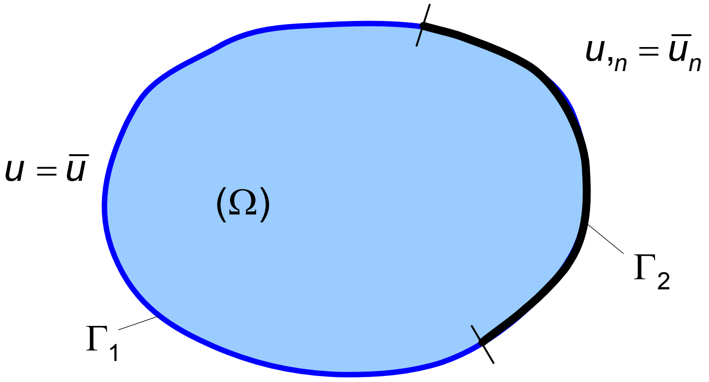
$$u = \bar{u} \quad \text{on} \quad \Gamma_1, \qquad u_{,n} = \bar{u}_{,n} \quad \text{on} \quad \Gamma_2 \tag{10b}$$

where $\Gamma_1 \cup \Gamma_2 = \Gamma$ and $\Gamma_1 \cap \Gamma_2 = \{\varnothing\}$

Greens's adjoint problem

$$\nabla^2 G = \delta(Q-P) \quad \text{in} \quad \Omega \tag{11a}$$

$$G = 0 \quad \text{on} \quad \Gamma_1, \qquad G_{,n} = 0 \quad \text{on} \quad \Gamma_2 \tag{12b}$$



Domain with mixed BC's

Choose u satisfying BVP (10a,b)

Choose $v = G(Q,P)$ satisfying BVP (10b)

and apply Green's identity (4)

$$\int_\Omega \left[G\nabla^2 u - u\delta(Q-P) \right] d\Omega = \int_\Gamma \left(G\frac{\partial u}{\partial n} - u\frac{\partial G}{\partial n} \right) ds \tag{13}$$

Solution

$$u(P) = \int_{\Gamma_1} \left[\bar{u}(q) G_{,n}(P,q) \right] ds_q - \int_{\Gamma_2} \left[G(P,q)\bar{u}_{,n}(q) \right] ds_q \tag{14}$$

Analytical solutions of the Green’s BVP is feasible ONLY for domains of simple geometry.

There is extended literature on the subject, see

[2] Greenberg, M., 1971. *Application of Green’s Functions in Science and Engineering*, Prentice Hall, Englewood Cliff, New Jersey.

[3] Roach, G.F., 1970. *Green’s Functions*, Van Nostrand Reinhold Company, London.

For domains of ARBITRARY shape an analytical solution is out of Question.

Numerical solutions have been also proposed, e.g.

Katsikadelis J.T. and Sapountzakis E.J. (1986) [5]

Katsikadelis J.T. and Sapountzakis E.J. (1987) [6]

Katsikadelis J.T. and Nerantzaki M.S. (1988) [7]

Nevertheless, the method of singularities is NOT EFFICIENT for realistic engineering problems

2. Fredholm's method.

Erik Ivar Fredholm

Born: 7 April 1866 in Stockholm, Sweden
Died: 17 Aug 1927 in Danderyd, County of Stockholm, Sweden

In 1872, Betti presented a general method for integrating the equations of elasticity and deriving their solution in integral form. Basically, this may be regarded as a direct extension of Green's approach to the Navier equations of elasticity.

In 1885, Somigliana used Betti's reciprocal theorem to derive the integral representation of the solution for the elasticity problem, including in its expression the body forces, the boundary displacements and tractions.

The fatherhood, however, of the Boundary Element Method could be attributed to Fredholm. In 1903 he was the first to use singular boundary integral equations to establish not specified boundary quantities in BVP for the Laplace equation.

$\nabla^2 u = 0$ in Ω

$u = \bar{u}$ on Γ Dirichlet BC

$u_{,n} = \bar{u}_{,n}$ on Γ Neumann BC

$$\int_\Omega \left(v\nabla^2 u - u\nabla^2 v \right) d\Omega = \int_\Gamma \left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n} \right) ds \qquad (1)$$

Choose u to satisfy the Laplace equation

Choose $v = v(P,Q)$ to be a particular solution of the equation

$$\nabla^2 v = \delta(Q - P) \qquad (2)$$

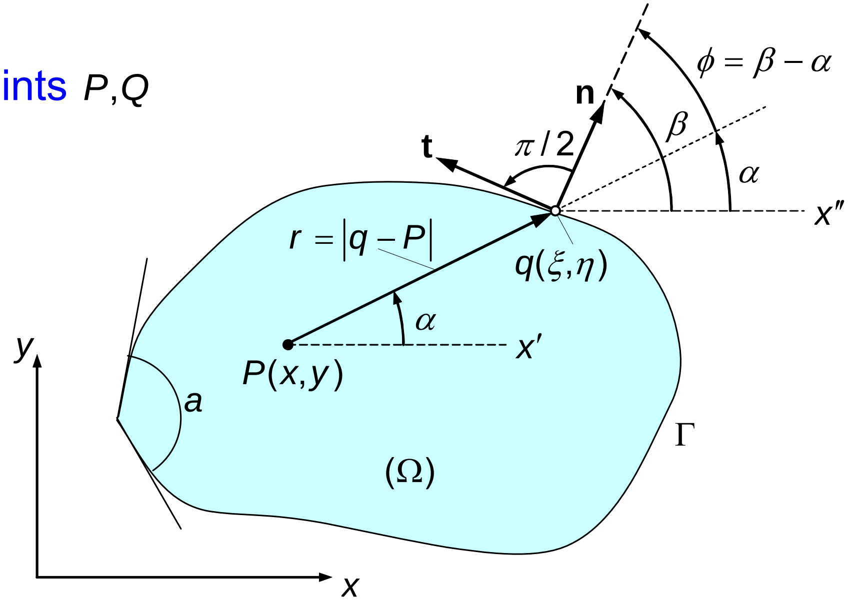
then

$$u(P) = -\int_\Gamma \left(v(P,q)\frac{\partial u(q)}{\partial n_q} - u\frac{\partial v(P,q)}{\partial n_q} \right) ds_q \qquad (3)$$

The particular solution of (2) can be established and is given as

$$v = \frac{1}{2\pi}\ln r, \qquad r = \|P - Q\| \text{ Euclidean distance between points } P,Q$$

$$\frac{\partial v}{\partial n_q} = \frac{1}{2\pi}\frac{r_{,n}}{r}, \qquad r_{,n} = \cos\phi$$



$v = v(P,Q) = v(Q,P)$ is the FUNDAMENTAL SOLUTION of the governing equation

We let now $P \to p \in \Gamma \quad r = 0$. Singular integrals

$$\lim_{P\to p}\int_\Gamma u_{,n}\ln r\, ds = \int_\Gamma u_{,n}\ln r\, ds$$ single layer potential. It varies continuously

$$\lim_{P\to p}\int_\Gamma u\frac{1}{2\pi}\frac{r_{,n}}{r}ds = \frac{1}{2\pi}\int_\Gamma u\frac{r_{,n}}{r}ds + \frac{a}{2\pi}u$$ double layer potential. It varies discontinuously

Thus (3) becomes

$$\frac{a}{2\pi}u(p) = -\int_{\Gamma}\left(v(P,q)\frac{\partial u(q)}{\partial n_q} - u\frac{\partial v(P,q)}{\partial n_q}\right)ds_q \qquad (4)$$

For points p where the boundary is smooth $a = \pi$

$$\boxed{\frac{1}{2}u = -\int_{\Gamma}(vu_{,n} - uv_{,n})ds \qquad (5)}$$

Eq. (5) is a COMPATIBILITY RELATION for u and $u_{,n}$ on Γ

Or a SINGULAR BOUNDARY INTEGRAL EQUATION for the not specified boundary quantity

Except for very simple geometries (e.g. circle) an analytical solution of (5) is out of Question.

Fredholm used (5) only to determine the necessary BC’s for the Potential problem (Laplace equation)

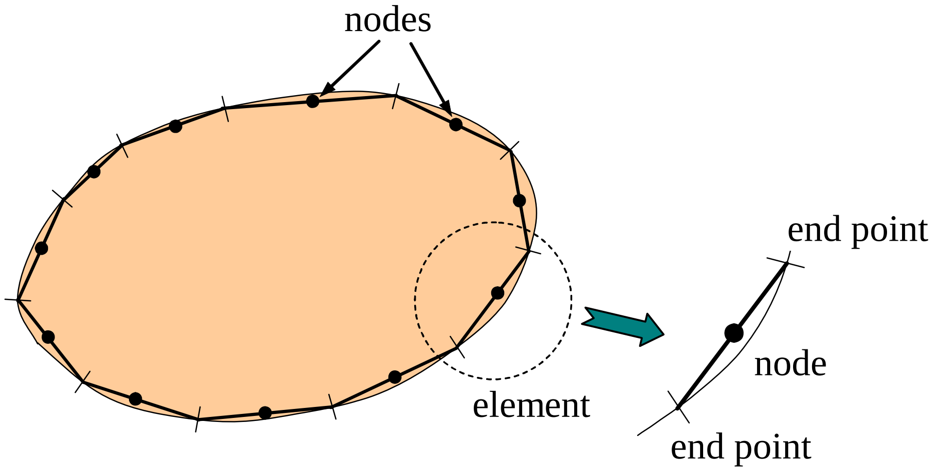
The Boundary Element Method. BEM [1]

The BEM is the BIEM with boundary discretization for numerical solution of the singular boundary integral equation

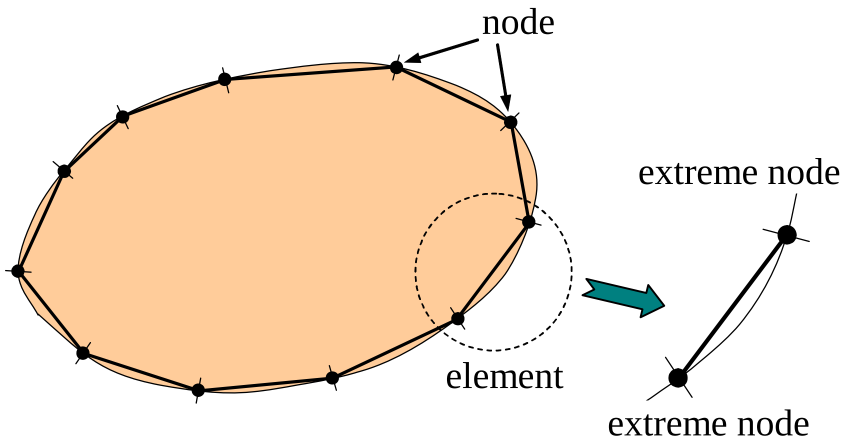
The first works that laid the foundation of BEM as a computational technique for solving PDEs appeared in the early sixties with the advent of the computers.

Jaswon and Symm used Fredholm's equations to solve some two-dimensional problems of potential theory

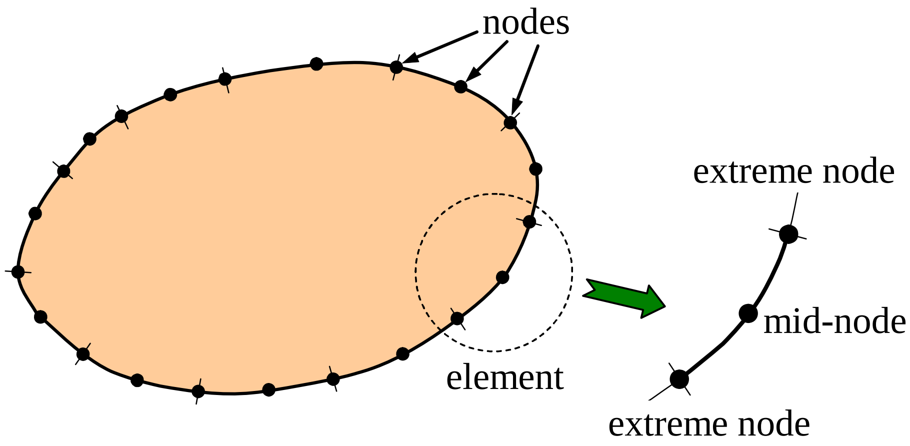
Figure 4.1 Various types of boundary elements.



(a) Constant elements



(b) Linear elements



(c) Parabolic elements

$$\frac{a}{2\pi}u(p) = -\int_{\Gamma}\left(v(P,q)\frac{\partial u(q)}{\partial n_q} - u\frac{\partial v(P,q)}{\partial n_q}\right) ds_q \qquad (4)$$

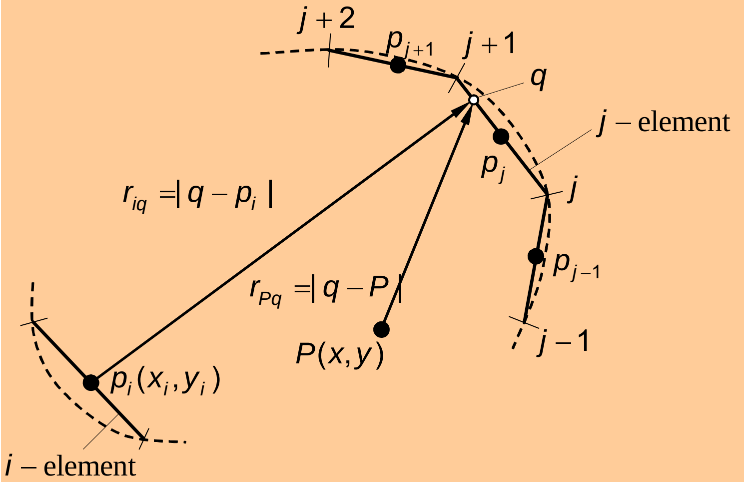
$$\frac{a}{2\pi}u^i = -\sum_{j=1}^{N}\int_{\Gamma_j} v(p_i,q)\frac{\partial u(q)}{\partial n_q}ds_q + \sum_{j=1}^{N}\int_{\Gamma_j} u(q)\frac{\partial v(p_i,q)}{\partial n_q}ds_q \qquad (4.1)$$

For constant elements, the boundary is smooth at the nodal points, hence $a = \pi$ and we have

$$-\frac{1}{2}u^i + \sum_{j=1}^{N}\left(\int_{\Gamma_j}\frac{\partial v}{\partial n}ds\right)u^j = \sum_{j=1}^{N}\left(\int_{\Gamma_j} v ds\right)u_n^j \qquad (4.2)$$

$$-\frac{1}{2}u^i + \sum_{j=1}^{N}\hat{H}_{ij}u^j = \sum_{j=1}^{N}G_{ij}u_n^j \qquad (4.4)$$

$$H_{ij} = \hat{H}_{ij} - \frac{1}{2}\delta_{ij}, \quad \hat{H}_{ij} = \int_{\Gamma_j}\frac{\partial v(p_i,q)}{\partial n_q}ds, \quad G_{ij} = \int_{\Gamma_j} v(p_i,q)ds \qquad (4.5)$$



$$[H]\{u\} = [G]\{u_n\} \qquad (4.7)$$

$[H]$, $[G]$ are $N \times N$ square matrices,

$\{u\}$ and $\{u_n\}$ are vectors of dimension N.

Applying the BC’s and separating the unknown boundary quantities from the specified ones and rearranging

$$[A]\{X\}=\{B\} \qquad (4.9)$$

which yields the unspecified $\{u\}$ and $\{u_n\}$.

$[A]$ is invertible except for NEUMANN PROBLEM. Gauss elimination works very well

Then the solution at any point $P(x,y)\in\Omega$ is obtained from the discretized countered part of the integral representation, i.e.

$$u(P)=\sum_{j=1}^{N}\hat{H}_{ij}u^{j}-\sum_{j=1}^{N}G_{ij}u_{n}^{j} \qquad (4.11)$$

and the derivatives

$$u_{,x}(P)=\left(\frac{\partial u}{\partial x}\right)_{P}=\sum_{j=1}^{N}\left(\hat{H}_{Pj}\right)_{,x}u^{j}-\sum_{j=1}^{N}\left(G_{Pj}\right)_{,x}u_{n}^{j} \qquad (4.15)$$

$$u_{,y}(P)=\left(\frac{\partial u}{\partial y}\right)_{P}=\sum_{j=1}^{N}\left(\hat{H}_{Pj}\right)_{,y}u^{j}-\sum_{j=1}^{N}\left(G_{Pj}\right)_{,y}u_{n}^{j} \qquad (4.16)$$

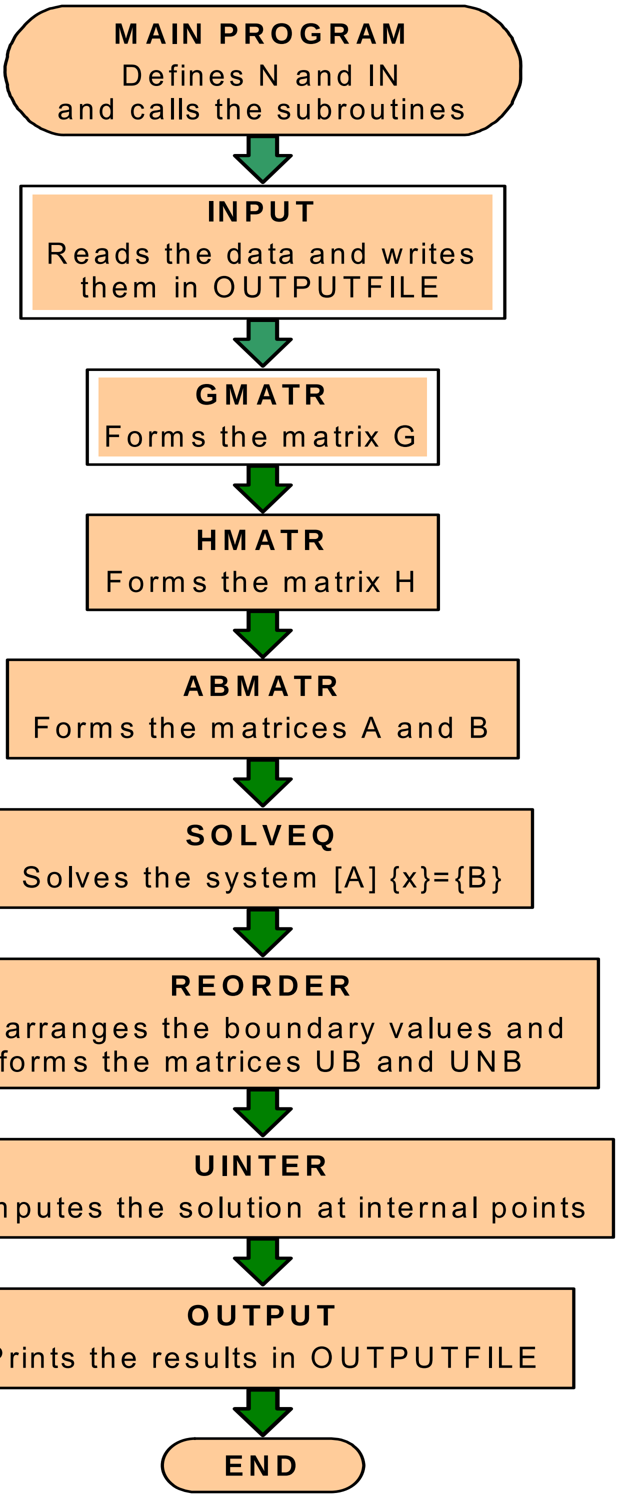


Figure 4.7 Macro flow chart of program LABECON.

The BEM for the Poisson equation

Poisson's equation

$\nabla^2 u = f \quad \text{in} \quad \Omega$

$u = \bar{u} \quad \text{on} \quad \Gamma$ Dirichlet BC

$u_{,n} = \bar{u}_{,n} \quad \text{on} \quad \Gamma$ Neumann BC

$$\int_\Omega \left(v\nabla^2 u - u\nabla^2 v \right) d\Omega = \int_\Gamma \left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n} \right) ds \qquad (3.1)$$

Choose u to satisfy the Poisson equation

Choose $v = v(P,Q)$ satisfying $\nabla^2 v = \delta(Q-P)$

Then the integral representation of the solution becomes

$$u(P) = \int_\Omega vf\, d\Omega - \int_\Gamma \left(vu_{,n} - uv_{,n} \right) ds \qquad (3.2)$$

and the boundary integral equation

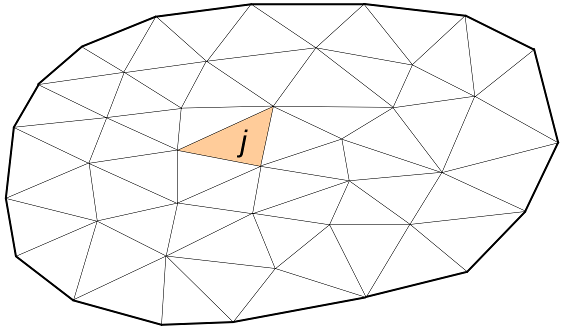
$$\frac{a}{2\pi} u(P) = \int_\Omega vf\, d\Omega - \int_\Gamma \left(vu_{,n} - uv_{,n} \right) ds \qquad (3.3)$$

New problem: EVALUATION OF THE DOMAIN INTEGRAL

Evaluation of the domain integrals

(a). Use domain discretization

$$\int_{\Omega} vf \, d\Omega = \sum_j \int_j vf \, d\Omega_j \qquad (3.51$$



Discretization of Ω in triangular cells

The BEM looses its pure boundary characters, hence its advantages over domain methods

(b). Conversion of domain integral into boundary line integral

b1. Exact method for **2D domains.**

The function f is arbitrary defined in $\Omega^* \subset \Omega$.

We first establish another function F which satisfies the following Poisson equation

$$\nabla^2 F = f \qquad (3.52)$$

The Green identity is then applied for the functions v and F in the domain Ω^*,

$$\int_{\Omega^*} \left(v\,\nabla^2 F - F\,\nabla^2 v \right) d\Omega = \int_{\Gamma^*} \left(v\frac{\partial F}{\partial n} - F\frac{\partial v}{\partial n} \right) ds$$

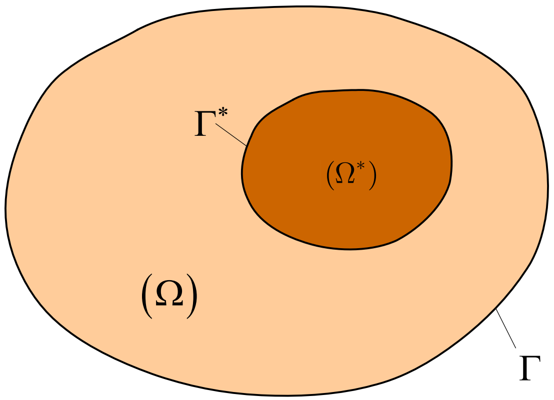
which gives

$$\int_{\Omega^*} vf\,d\Omega = \int_{\Omega^*} F\,\delta(Q-P)\,d\Omega_Q + \int_{\Gamma^*} \left(v\frac{\partial F}{\partial n} - F\frac{\partial v}{\partial n} \right) ds \qquad (3.53)$$

We distinguish two cases:

(i) $\Omega^* \equiv \Omega$. The point P is always inside Ω and Eq. (3.53) yields

$$\int_{\Omega} vf\,d\Omega = F + \int_{\Gamma} \left(v\frac{\partial F}{\partial n} - F\frac{\partial v}{\partial n} \right) ds \qquad (3.54)$$



Domain Ω^*ÍΩ

(ii) $\Omega^* \subset \Omega$. The point P may be inside Ω^*, on Γ^*, or outside Ω^*. In this case Eq. (3.53) gives

$$\int_{\Omega^*} vf\, d\Omega = \varepsilon(P)F + \int_{\Gamma^*}\left(v\frac{\partial F}{\partial n} - F\frac{\partial v}{\partial n}\right)ds \qquad (3.55)$$

where $\varepsilon(P) = 1, \frac{1}{2}, 0$ depending on whether the point P is inside Ω^*, on Γ^* or outside Ω^*, respectively.

Evaluation of F from f

$$z = x + iy\,, \qquad \bar{z} = x - iy \qquad \left(i = \sqrt{-1}\right) \qquad (3.41a)$$

and its inverse is

$$x = \frac{z+\bar{z}}{2}\,, \qquad y = \frac{z-\bar{z}}{2i} \qquad (3.41b)$$

We can readily show that $\nabla^2 F = f$ is transformed to

$$4\frac{\partial^2 F}{\partial z\, \partial \bar{z}} = f(z,\bar{z}) \qquad (3.42)$$

which is readily integrated to yield

$$F(z,\bar{z}) \quad \Rightarrow \quad F(x,y)$$

Solution of $\nabla^2 F = f$ for **3D DOMAINS** ? (Quaternions?).

b2. Approximate methods for **2D and 3D domains using Radial Basis Functions (RBFs)**.

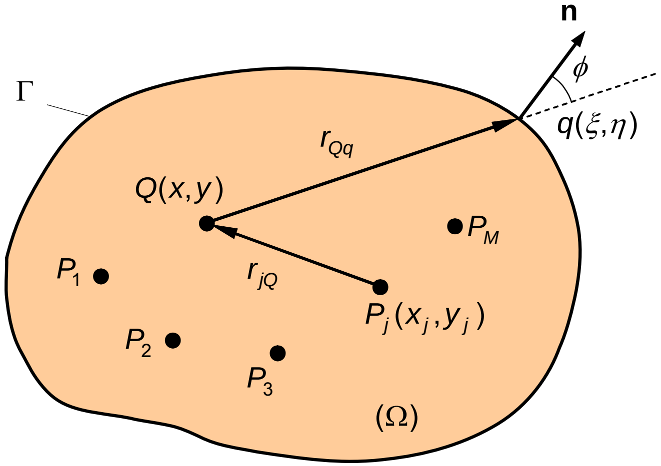
The Dual Reciprocity Method

Function f is approximated

$$f(Q)=\sum_{j=1}^{M} a_j\phi_j(r_{jQ}) \qquad Q:(x,y) \qquad (4.34)$$

r_{jQ} : is the distance between the field point $Q(x,y)$ and the collocation point $P_j(x_j,y_j)$.

$$r_{jQ}=\left|Q-P_j\right|=\sqrt{(x-x_j)^2+(y-y_j)^2} \qquad (4.35)$$



The value of f at collocation point i is

$$f_i=f(P_i)=\sum_{j=1}^{M} a_j\,\phi_j(r_{ji}) \qquad (i=1,2,\ldots,M) \qquad (4.36)$$

where

$$r_{ji}=\left|Q_i-Q_j\right|=\sqrt{(x_i-x_j)^2+(y_i-y_j)^2} \qquad (4.37)$$

Using matrix notation Eq. (4.36) is written as

$$\{f\}=[\Phi]\{a\} \qquad (4.38)$$

$\{f\}$: is a vector containing the values of function f at the M collocation points,

$\{a\}$:is the vector of the M unknown coefficients and

$[\Phi]=[\phi_j(r_{ji})]$ an $M\times M$ known matrix.

Assuming that $[\Phi]$ is not singular, Eq. (4.38) can be solved to yield

$$\{a\}=[\Phi]^{-1}\{f\} \qquad (4.39)$$

The integral

$$I(P)=\int_{\Omega} v(P,Q)f(Q)d\Omega_Q\,, \quad P(x,y)\in(\Omega\cup\Gamma),\ Q(x,y)\in\Omega \quad (4.33)$$

is written as

$$I(P)=\sum_{j=1}^{M} a_j\left[\int_{\Omega} v(P,Q)\phi_j(r_{jQ})d\Omega_Q\right] \qquad (4.40)$$

We apply Green’s reciprocal identity

$$\int_{\Omega}\left(v\nabla^2 u-u\nabla^2 v\right)d\Omega=\int_{\Gamma}\left(v\frac{\partial u}{\partial n}-u\frac{\partial v}{\partial n}\right)ds$$

for v satisfying

$$\nabla^2 v=\delta(Q-P)$$

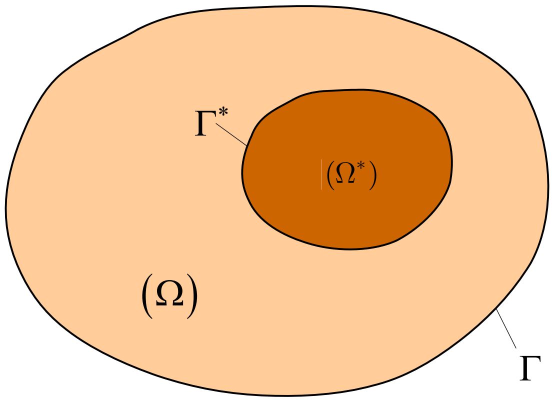
and $u=\hat{u}(r)$ satisfying

$$\nabla^2\hat{u}_j=\phi_j(r) \qquad (4.41)$$

$$\int_{\Omega^*} v(P,Q)\phi_j(Q)\,d\Omega_Q = \int_{\Omega^*} v(P,Q)\nabla^2\hat{u}_j(Q)\,d\Omega_Q$$

$$= \varepsilon(P)\hat{u}_j(P)$$

$$+\int_{\Gamma^*}\left[v(P,q)\frac{\partial \hat{u}_j(q)}{\partial n_q} - \hat{u}_j(q)\frac{\partial v(P,q)}{\partial n_q}\,ds_q\right] \qquad (4.42)$$



Domain $\Omega^* \subset \Omega$

where $\varepsilon(P) = 1, \tfrac{1}{2}, 0$ depending on whether P is inside Ω^*, on Γ^* or outside Ω^*, respectively.

Substituting the domain integral of Eq. (4.42) into Eq. (4.40), the latter becomes

$$I(P) = \sum_{j=1}^{M} a_j \left\{\varepsilon(P)\hat{u}_j(P) + \int_{\Gamma^*}\left[v(P,q)\frac{\partial \hat{u}_j(q)}{\partial n_q} - \hat{u}_j(q)\frac{\partial v(P,q)}{\partial n_q}\,ds_q\right]\right\} \qquad (4.43)$$

The RBFs in R^n. Scattered data interpolation [8]

Let $x_1, x_2, \ldots x_M \in \Omega \subset \mathbf{R}^n$ given set of nodes and

$f_j(x) = f\left(\left\| x - x_j \right\|\right) \notin \mathbf{R}, \quad j = 1, 2, \ldots, M$ a set of RBF functions

$\left\| x - x_j \right\|$ is the Euclidean distance

$u_1, u_2, \ldots, u_M \notin \mathbf{R}$ given interpolation values at data locations $x_1, x_2, \ldots x_M \in \Omega \subset \mathbf{R}^n$

The RBF interpolant

$$u(x) = \sum_{j=1}^{M} a_j f_j(x)$$

is obtained by solving the system of M linear equations

$$\sum_{j=1}^{M} a_j f_j(x_i) = u_i, \quad i = 1, 2, \ldots, M$$

Commonly used Radial Basis Functions

There is an infinite class of RBFs. The most commonly used are

Thin plate Splines $\phi_j = r^2 \ln r \,,$

Polynomial $\phi_j = 1 + r + \cdots + r^m$

No adjustable parameters
Uniform scaling

Gaussians $\phi_j = \exp\left(-\frac{r}{c_j^2}\right)$

Multiquadrics $\phi_j = \sqrt{r^2 + c_j^2}$

Adjustable parameter c_j

Produces local dilatation

$r = \left\| x - x_j \right\|$ The Euclidean distance between field point x and collocation point x_j

Particular solution of the Poisson Equation with RBF source

$$\nabla^2 \hat{u}_j = \phi_j(r)$$

when $\phi_j(r) = \sqrt{c^2 + r^2}$

In 2D

$$\nabla^2 \hat{u}_j = \frac{1}{r}\frac{d}{dr}(r\frac{d\hat{u}_j}{dr}) = \left(r^2 + c^2\right)^{1/2}$$

$$\hat{u}_j = \frac{1}{9}f^3 + \frac{1}{3}fc^2 - \frac{c^3}{3}\ln(c+f) + G\ln r + F$$

In 3D

$$\nabla^2 \hat{u}_j = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\hat{u}}{dr)}\right) = \left(r^2 + c^2\right)^{1/2}$$

$$\hat{u}_j = \frac{c^4}{8r}\ln\left(\frac{r+f}{c}\right) + \frac{\left(2f^2 + 3c^2\right)f}{24} - \frac{G}{r} + F$$

The BEM for Domains with multiple boundaries [1]

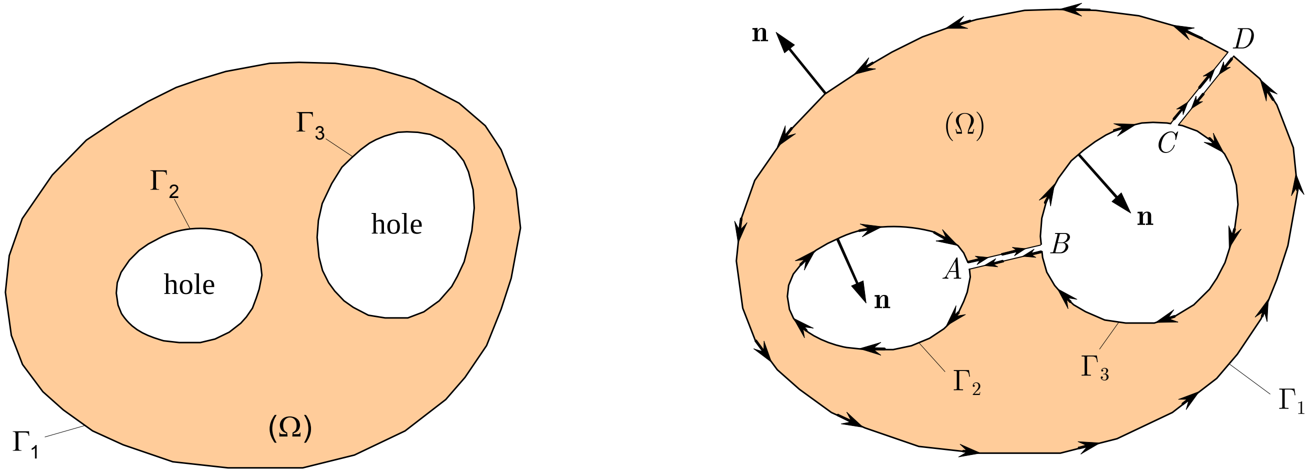


Figure 4.15 Multiply connected domain Ω.

The Green's identity is also valid

$$\int_{\Omega}\left(v\nabla^2 u - u\nabla^2 v\right)d\Omega = \int_{\Gamma}\left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\right)ds \qquad (4)$$

The Ingredients of the BIE Method

Given the Linear BVP:

$$L(u) = g(x) \qquad x \in \Omega$$

$$B(u) = \bar{g}(x) \qquad x \in \Gamma$$

1. Establish a Green's reciprocal identity for the linear operator $L(\cdot)$ of order n

   $\int_\Omega \left(vL(u) - uL^*(v)\right) d\Omega = \int_\Gamma M(u,v)ds$ This always possible

2. $M(u,v)$ is a bilinear differential expression of order $n-1$. $L^*(v)$ is the adjoint operator

   if $L^*(.) = L(.)$ self-adjoint

3. Establish the fundamental solution of

   $L^*(v) = \delta(x-\xi)$ Possible only for a few simple cases.

4. Establish the integral representation of the solution

   $u = \int_\Omega vg(x)d\Omega - \int_\Gamma M(u,v)ds$

5. Establish the boundary integral equations to apply the BCs
6. Solve the boundary Integral equations numerically to obtain the not specified boundary quantities
7. Obtain the solution from its integral representation

MAJOR DRAWBACKS OF THE BEM

A. For Linear PDEs

We consider the Complete 2nd Order elliptic PDE

$$L(u) = A\frac{\partial^2 u}{\partial x^2} + 2B\frac{\partial^2 u}{\partial x\,\partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G \qquad (2.19)$$

$A, B, \ldots, F$ are given functions of x and y in Ω satisfying the ellipticity condition $B^2 - AC < 0$

We consider the integral

$$\int_\Omega v\,L(u)\,d\Omega \qquad (2.20)$$

$v = v(x, y)$ twice continuous differentiable

Integrating Eq. (2.20) by parts repeatedly until all derivatives of u are eliminated we arrive at

$$\int_\Omega \left\{ v\,L(u) - u\,L^*(v) \right\} d\Omega = \int_\Gamma \left(X\,n_x + Y\,n_y \right) ds \qquad (2.21)$$

where

$$X = A\left(v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x}\right) + B\left(v\frac{\partial u}{\partial y} - u\frac{\partial v}{\partial y}\right) + \left(D - \frac{\partial A}{\partial x} - \frac{\partial B}{\partial y}\right)uv \qquad (2.22)$$

$$Y = B\left(v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x}\right) + C\left(v\frac{\partial u}{\partial y} - u\frac{\partial v}{\partial y}\right) + \left(E - \frac{\partial B}{\partial x} - \frac{\partial C}{\partial y}\right)uv \qquad (2.23)$$

$$L^*(v) = \frac{\partial^2(Av)}{\partial x^2} + 2\frac{\partial^2(Bv)}{\partial x\partial y} + \frac{\partial^2(Cv)}{\partial y^2} - \frac{\partial(Dv)}{\partial x} - \frac{\partial(Ev)}{\partial y} + Fv \qquad (2.24)$$

$L^*(v)$ is self-adjoint if $L^*(v) = L(v)$

If we take v to satisfy

$$L^*(v) = \delta(Q-P) \qquad (2.26)$$

then we obtain the integral representation of the solution

$$u = \int_\Omega vG\,d\Omega - \int_\Gamma \left(X\,n_x + Y\,n_y\right)ds$$

- The reciprocal identity can be always established integrating by parts.

However, Special care should be paid in order that the boundary terms can express physical boundary quantities, which is necessary to apply the BCs.

This is simple for self-adjoint operators, where Betti's reciprocal theorem can be applied.

MAJOR PROBLEM: Is it possible to establish the fundamental solution of the adjoint equation?

Unless for special cases, of the coefficients (e.g. constant coefficients) the problem can not be solved.

Even in cases where the fundamental solution can be established it may difficult to treat it analytically and/or numerically [4a,b].

Hence, The BEM, this elegant and efficient method, practically does not work.

B. For Nonlinear PDEs

$$N(u)=g(\mathbf{x}), \qquad \mathbf{x}:\{x,y\}\in\Omega$$

$$B(u)=b(\mathbf{x}), \quad \mathbf{x}:\{x,y\}\in\Gamma$$

$N(u)$, $B(u)$ Nnonlinear operators

- A reciprocal identity does not exist for nonlinear operators
- No Fundamental solution
- No integral representation

THEREFORE

the BEM can not be developed for nonlinear partial differential equations.

QUESTION

Is it possible to develop the BEM for general
Linear and Nonlinear PDEs?

The last 25 years extended research has been done
to answer
this question and is still ongoing

BEM Solution Methods for Linear and Nonlinear PDEs
Using Known Fundamental solutions

The idea is the following

Extract a DOMINANT LINEAR OPERATOR $T(u)$ from $N(u)$ with a

KNOWN FUNDAMENTAL SOLUTION,

IF POSSIBLE !!,

and shift the remaining terms in the right hand side as body forces, i.e.

$$N(u) = g(\mathbf{x}), \qquad \mathbf{x}:\{x,y\} \in \Omega$$

$$T(u) = b(u,\mathbf{x})$$

$$b(u,\mathbf{x}) = N(u) - T(u) + g(\mathbf{x})$$

The order of $b(u,\mathbf{x})$ must be less than that of $T(u)$

Example #1 Temperature distribution

$$u_{xx} + u_{yy} - k(\mathbf{x})(1 + u_x^2 + u_y^2)^{3/2} = g(\mathbf{x})$$

$$\nabla^2 u = b(u, \mathbf{x})$$

$$b(u, \mathbf{x}) = k(\mathbf{x})(1 + u_x^2 + u_y^2)^{3/2} + g(\mathbf{x})$$

Example #2 Plate with combined inplane action

$$D\nabla^4 w - (N_x w,_{xx} + 2N_{xy} w,_{xy} + N_y w,_{yy}) + f_x w,_x + f_y w,_y = g(\mathbf{x}) \qquad \text{in } \Omega$$

$$D\nabla^4 w = b(w, \mathbf{x})$$

$$b(w, \mathbf{x}) = (N_x w,_{xx} + 2N_{xy} w,_{xy} + N_y w,_{yy}) - f_x w,_x - f_y w,_y + g(\mathbf{x})$$

In these examples the dominant operators are $T = \nabla^2 \quad and \quad T = \nabla^4$.

Both have a known reciprocal identity and a fundamental solution, thus an integral representation, i.e

$$u = -\int_\Omega vg \, d\Omega - \int_\Gamma (vu,_n - uv,_n) ds$$

$$w(\mathbf{x}) = \int_\Omega vb d\Omega + \int_\Gamma (vVw + w,_n Mv - v,_n Mw - wVv) ds - \sum_k \left(v [\![Tw]\!] - w [\![Tv]\!] \right)_k$$

Then the problem can be solved using

- Domain Boundary Element Method (D/BEM)
- Dual Reciprocity Method (DRM) [9] as boundary-only method, with meshless domain nodal points

However, These methods cannot be employed if [10]

1. The differential operator cannot be put in the form $T(u)=b(u,\mathbf{x})$, e.g

Example #3 Surface with specified mean curvature

$$(1+u_y^2)u_{xx}-2u_xu_yu_{xy}+(1+u_x^2)u_{yy}-k(x,y)(1+u_x^2+u_y^2)^{3/2}=0$$

$$\nabla^2 u=b(u,\mathbf{x})$$

$$b(u,\mathbf{x})=-u_y^2u_{xx}-u_x^2u_{yy}+2u_xu_yu_{xy}+k(x,y)(1+u_x^2+u_y^2)^{3/2}$$ No good

Example #4 Surface with specified Gaussian curvature

$$u_{xx}u_{yy}-u_{xy}^2=f(x,y)$$ Impossible to extract a linear operator

Major Drawbacks of these methods

The D/BEM	loses the advantages of a boundary method.
The DRM.	Although it avoids domain discretization and integration, it requires different formulations for different substitute body force as well as for different operators, that is each equation requires ad hoc special treatment Therefore different computer programs must be written for different PDEs of the same order It fails to apply to coupled PDEs, at least no such applications is reported in the literature.

The Analog Equation Method (AEM)

It overcomes all the above drawbacks and renders BEM a boundary-only powerful computational tool for solving difficult realistic problems of mathematical physics and engineering