THE COCKCROFT INSTITUTE
Autumn Lecture Courses, 2006.

Linear Dynamics for Particle Accelerators

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1 Aims

The goal will be to introduce students to the basics of linear beam dynamics in high-energy accelerator beamlines. Starting from a fundamental description of the dynamics of a relativistic particle in an electromagnetic field, the equations of motion in "linear" accelerator components (including drift spaces, dipoles, solenoids, quadrupoles and RF cavities) will be derived, and then solved using linear approximations (including the paraxial approximation). The effects of a curved coordinate system (in dipoles) will be considered. The physical interpretation of the solutions of the equations of motion, as betatron and synchrotron oscillations, will be discussed. A matrix representation of the linear dynamics will be explored, which, in turn, will lead to a description in terms of beta functions, dispersion, phase advances, and emittances. Action-angle variables will be introduced, and their applications discussed. Finally, the effects of linear imperfections on orbit and beam size will be considered.

By the end of the course, students should be able to:

- derive the Hamiltonian for relativistic particle motion in an accelerator, defining appropriate variables;
- derive the equations of motion for a relativistic particle in "linear" accelerator components, including drift spaces, dipoles, solenoids, quadrupoles and RF cavities;
- solve the equations of motion for a relativistic particle in "linear" accelerator components using linear approximations;
- use the solutions for the equations of motion to write down the transfer matrices for "linear" accelerator components;
- explain the effects of a curved reference trajectory in dipoles (including dispersion and phase slip);
- explain the parameterisation of uncoupled transfer matrices using the Twiss parameters and phase advance;
- explain the important properties of simple periodic FODO beamlines;

- define action-angle variables and explain their significance;
- relate the matched beam distribution (sigma matrix) in an uncoupled lattice to the Twiss parameters and beam emittances;
- describe the generalisation of the uncoupled Twiss parameters, phase advances and beam emittances to the general coupled case in three degrees of freedom;
- describe the effects of lattice imperfections, including small steering and focusing errors.

The formal foundations in classical, relativistic mechanics will be used to develop a highly practical description of linear beam dynamics, in which the approximations used to find simple solutions to the equations of motion are made explicit. As a result of taking the course, students should be able to perform calculations and computations of the properties of linear beamlines, and should be prepared for courses covering more specialized and advanced topics, including nonlinear dynamics and collective effects.

2 Syllabus

The course will be presented in ten lectures, with homework problems (that will be graded) and tutorials. The first part of the course (lectures 1-5) will develop the dynamical description of particles in accelerators, based on the equations of motion for a relativistic charged particle in an electromagnetic field. The second part of the course (lectures 6-10) will develop beam optics from the linear transfer matrices derived in the first part, to describe the dynamics using lattice functions.

Part 1: Dynamics of a relativistic charged particle in the electromagnetic fields of an accelerator beamline.

1. Review of Hamiltonian mechanics.
    - Newtonian and Lagrangian mechanics.
    - Canonical momentum.
    - Hamiltonian mechanics and Hamiltons equations.
    - Hamiltonian for a charged particle in an electromagnetic field.
    - Symplecticity.
    - Canonical transformations; generating functions.
2. The accelerator Hamiltonian in a straight coordinate system.

- Hamiltonian for a relativistic charged particle in an electromagnetic field.
- Path length as the independent variable.
- Reference momentum.
- Energy deviation, and normalised momenta.
- Dynamical map for a drift space.
- Paraxial approximation and transfer matrix for a drift space.

3. The Hamiltonian for a relativistic particle in a general electromagnetic field using accelerator coordinates.
   - Hamiltonian in a curved coordinate system.
   - Magnetic multipole fields and multipole magnets.
   - Magnetic vector potential in curved coordinates.
   - Hamiltonian for a particle in the field of a dipole magnet.
   - Weak focusing.
   - Dispersion.
   - Dynamical map and transfer matrix for a dipole magnet.
4. Dynamical maps for linear elements.
   - Hamiltonian and transfer matrix for a normal quadrupole.
   - Hamiltonian and transfer matrix for a skew quadrupole.
   - Electromagnetic field in an RF cavity.
   - Hamiltonian and transfer matrix for an RF cavity.
   - Hamiltonian and transfer matrix for a solenoid.
   - Hamiltonian and transfer matrix for a combined-function bend.
   - A word about fringe fields.
5. Three loose ends: edge focusing; chromaticity; beam rigidity.
   - Dynamics and edge focusing in dipole fringe fields.
   - Transfer matrix for dipole fringe field with pole-face rotation.
   - Dynamics and transfer matrix for a solenoid fringe field.
   - Chromaticity. Energy deviation as a parameter.
   - Beam rigidity.

Part 2: Description of beam dynamics using optical lattice functions.

6. Linear optics in periodic, uncoupled beamlines.
   - The FODO lattice.
   - Transfer matrix for a FODO cell.
   - Phase-space ellipse from multiple passes through FODO cells.
   - Twiss parameters used to parameterise the transfer matrix.
   - Action-angle variables.
   - Twiss parameters used to parameterise the phase-space ellipse.
   - Evolution of the Twiss parameters: betatron functions.
   - Phase advance; betatron tunes.
7. Including longitudinal dynamics.
   - Longitudinal dynamics in a FODO beamline.
   - Transfer matrix for a FODO cell with dipoles.
   - The dispersion function.
   - Longitudinal dynamics in a storage ring.
   - Momentum compaction factor; phase slip factor.
   - Transition.
   - Longitudinal dynamics with RF cavities.
   - Synchrotron oscillations; phase stability.
   - Effects of RF curvature; longitudinal phase space and RF acceptance.
8. Bunches of many particles.
   - First- and second-order moments of a beam distribution.
   - Sigma matrix.
   - Emittances of a bunch distribution.
   - Matched distribution.
   - Twiss parameters used to parameterise a bunch distribution.
   - Acceleration: change in reference momentum; normalised emittance; adiabatic damping.
9. Coupled optics.
   - Properties of symplectic matrices.
   - The normalising transformation.

- Action-angle variables in three degrees of freedom.
- Eigenvalues of the transfer matrix and lattice tunes.
- Integer and half-integer resonances and linear dynamical stability.
- Matched distribution in three degrees of freedom
- Coupled lattice functions: beam distribution and emittances.

10. Effects of linear imperfections.
    - Quadrupole misalignments.
    - Steering errors and closed orbit distortion.
    - Orbit response matrix.
    - Sextupole misalignments.
    - Focusing errors: tune shifts and beta-beat.
    - Coupling errors.

3 Prerequisites

- Mathematics. Trigonometry: trigonometric identities; hyperbolic functions. Basic calculus: differentiation; integration; partial differentiation. Vector analysis. Linear algebra: matrix manipulation; eigenvalues and eigenvectors.
- Classical (Newtonian, Lagrangian and Hamiltonian) mechanics. A basic knowledge will be useful, but the material will be introduced without the assumption of prior expertise. It is strongly recommended to read in advance chapters from H. Goldstein, "Classical Mechanics", on Lagrangian mechanics, Hamiltonian mechanics, and canonical transformations. For a clear and concise summary of the necessary Hamiltonian mechanics, see: B. W. Montague, "Basic Hamiltonian Mechanics" (Proceedings of the CERN Accelerator School, 5$^{\text{th}}$ Advanced Accelerator Physics Course, Rhodes, Greece, 1993, CERN Yellow Report CERN-95-06).
- Electromagnetism. Students should be familiar with the electric scalar and magnetic vector potentials, and with gauge transformations of the magnetic vector potential. Students should know the relations between the potentials and the fields. Students should know that electromagnetic fields must be solutions of Maxwells equations, and should be able to demonstrate that given fields satisfy Maxwells equations in simple cases. A wide variety of texts covering the necessary material are available.

- Special relativity. Students should be familiar with the Lorentz transformations, with the relativistic relation between energy, mechanical momentum and mass, and with the use of the relativistic factors β and γ. A wide variety of texts covering the necessary material is available.